Proof— Setting in (1.1.10)

$$\varphi(x) = e^{i<x,k>}, \quad x \in \mathbb{R}^m,$$

with $k \in \mathbb{R}^m$, we obtain

$$\overline{\mu \circ T}\,(k) = e^{i<a,T^*k> - \frac{1}{2}<QT^*k,T^*k>}, \quad k \in \mathbb{R}^m.$$

$$= e^{i<Ta,k> - \frac{1}{2}<TQT^*k,k>}$$

By (1.1.8) we have

$$\overline{\mu \circ T}\,(k) = \overline{N(Ta, TQT^*)}\,(k), \quad k \in \mathbb{R}^m.$$

and the conclusion follows since the Fourier transform is one–to–one. ∎

1.2 Wiener measure

Se non si supponesse che le ω sono tali che ω(0) = 0, allora ω potrebbe assumere in 0 qualunque valore reale con ugual probabilità ⇒ si ottiene una distrib. di prob. equidistribuita su R ⇒ assurdo
$\Omega \sim \{\varphi : a + \omega,\ \omega \in C_0([0,T])$
$a \in \mathbb{R}\}$

We fix $T > 0$ and denote by $\Omega = C_0([0,T])$ the Banach space of all real continuous function on $[0,T]$, vanishing at 0, endowed with the norm

$$\boxed{\|\omega\| = \sup_{t \in [0,T]} |\omega(t)|, \quad \forall\, \omega \in \Omega.}$$

We denote by $\mathcal{F}$ the σ–algebra of all Borel subsets of Ω, that is the smallest σ–algebra of parts of Ω containing the open subsets of Ω.

We are going to construct the *Wiener probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We first define $\mathbb{P}$ on the cylindrical subsets of Ω.

A *cylindrical* subset of Ω is a set of the form

$$\boxed{I(t_1, \dots, t_n; B) = \{\omega \in \Omega : (\omega(t_1), \dots, \omega(t_n)) \in B\}}$$

where $n \in \mathbb{N}$, $0 \le t_1 < \dots < t_n \le T$ and $B \in \mathcal{B}(\mathbb{R}^n)$.

We denote by $\mathbb{K}$ the set of all cylindrical subsets of Ω. Notice that the cylindrical set $I(t_1, \dots, t_n; B)$ is not uniquely determined by $\boxed{t_1, \dots, t_n}$ and B. For instance, if $t_n < t_{n+1} \le T$ we have

$$I(t_1, \dots, t_n; B) = I(t_1, \dots, t_n, t_{n+1}; B \times \mathbb{R}).$$

(*) Moreover it is clear that if $F, G \in \mathbb{K}$ then there exist $n \in \mathbb{N}$, $0 \le t_1 < \dots < t_n \le T$, $A, B \in \mathcal{B}(\mathbb{R}^n)$, such that

$$F = I(t_{1_f}, \dots, t_{n_f}; A), \quad G = I(t_{1_g}, \dots, t_{n_g}; B).$$

Se $F = I(t_{1f}, \dots, t_{nf}; A)$ $\quad F = I(t_{1f}, \dots, t_{nf}, t_{1g}, \dots, t_{ng}; A \times \mathbb{R}^{n})$

$G = I(t_{1g}, \dots, t_{ng}, B)$ $\quad G = I(t_{1f}, \dots, t_{nf}, t_{1g}, \dots, t_{ng}; \mathbb{R}^n \times$

$F \cup G = (F \triangle G) \vee (F \cap G)$

$F \cap G = I(s_1, \dots, s_n; A \cap B)$

From this fact it follows immediately that $\mathbb{K}$ is an algebra. Moreover it is easy to check that the σ–algebra generated by $\mathbb{K}$ coincides with $\mathcal{F}$. For this it suffices to show that the closed balls

$$\boxed{\overline{B(a,r)} = \{\omega \in \Omega : \ \|\omega - a\| \le r\}, \ a \in \Omega, r > 0,}$$

insieme cilindrico

$\overline{B(a,r)} = \bigcap_{t\in[0,T]} \{\omega \in \Omega \,|\, |\omega(t) - a(t)| \le r\}$

belong to the σ–algebra generated by $\mathbb{K}$, and this follows from the equality [1]

$$\boxed{\overline{B(a,r)} = \{\omega \in \Omega : \ |\omega(\rho) - a(\rho)| \le r, \ \forall \rho \in \mathbb{Q} \cap [0,T]\}.}$$

a continua, quindi

$\overline{B(a,r)} = \bigcap_{q\in[0,T]\cap\mathbb{Q}} \{|\omega(q) - a(q)| \le r\}$

per avere una intersezione numerabile

Now we define a function $\mathbb{P} : \mathbb{K} \to [0,1]$ setting

$$\boxed{\begin{aligned} \mathbb{P}(I(t_1,...,t_n;B)) &= [(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})]^{-1/2} \\ &\times \int_B e^{-\frac{1}{2}\left[\frac{\xi_1^2}{t_1}+\cdots+\frac{(\xi_n-\xi_{n-1})^2}{t_n-t_{n-1}}\right]} d\xi_1\cdots d\xi_n, \end{aligned}} \quad (1.2.1)$$

where $n \in \mathbb{N}$, $0 \le t_1 < ... < t_n \le T$ and $B \in \mathcal{B}(\mathbb{R}^n)$.

It is easy to see that this definition does not depend on the choice of the particular representation of $I(t_1,...,t_n;B)$. For instance if $t_n < t_{n+1} \le T$, we have [2]

$$\begin{aligned} &\mathbb{P}(I(t_1,...,t_n,t_{n+1}; B\times\mathbb{R})) \\ &= \mathbb{P}(I(t_1,...,t_n;B)) \frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{(\xi_{n+1}-\xi_n)^2}{t_{n+1}-t_n}} d\xi_{n+1} \\ &= \mathbb{P}(I(t_1,...,t_n;B)). \end{aligned}$$

Da (*) $\Gamma_1 = I(t_1,..,t_n;B)$
$\Gamma_2 = I(t_1,..,t_n;A)$
$A\cap B = \emptyset$

It is easy to check that $\mathbb{P}$ is additive:

$$\mathbb{P}\left(\bigcup_{k=1}^n \Gamma_k\right) = \sum_{k=1}^n \mathbb{P}(\Gamma_k).$$

da cui
$\int_{\Gamma_1\cup\Gamma_2} = \int_{\Gamma_1} + \int_{\Gamma_2}$
e per i t_i si è a posto.

for all $n \in \mathbb{N}$, and for all mutually disjoint subsets $\Gamma_1,...,\Gamma_n$ in $\mathbb{K}$.

The following result is basic, for the proof see e.g. K. Itô and H. P. McKean [4], page 14.

---

[1] $\mathbb{Q}$ represents the set of all rational numbers.

[2] In the general case use the fact that setting $\boxed{g_t(x) = (2\pi t)^{-1/2} e^{-\frac{x^2}{2t}}}$, $t > 0, x \in \mathbb{R}$, one has $\boxed{g_t * g_s = g_{t+s}}$ for $\boxed{t > 0, s > 0}$ (†)

(Nel caso $t_{j-1} < t_{n+1} < t_j$)

(†) $\int \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{2s}} dx = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2s}} \int \frac{dx}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}\left(\frac{t+s}{t}\right) + \frac{2xy}{2s}}$

$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2s}\left(1-\frac{t}{t+s}\right)} \int \frac{dx}{\sqrt{2\pi s}} e^{-\frac{1}{2s}\left(x\sqrt{\frac{t+s}{t}} - y\sqrt{\frac{t}{t+s}}\right)^2} = \frac{1}{\sqrt{2\pi t}} \sqrt{\frac{t}{t+s}}\, e^{-\frac{y^2}{2(t+s)}} \int \frac{d\xi}{\sqrt{2\pi s}} e^{-\frac{\xi^2}{2s}}$

$= \frac{1}{\sqrt{2\pi(t+s)}} e^{-\frac{y^2}{2(t+s)}} \Rightarrow \mathbb{P}(I(t_1,...,t_j,t_{n+1},...,t_n))$

(1) $N(0, Q_{t_1,\dots,t_n})(B) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \int_B e^{-\frac12 <Q^{-1}x, x>} dx$

$\gamma \circ \mathbb{P}(B) = \mathbb{P}(\gamma^{-1}(B)) = \mathbb{P}(I(t_1,\dots,t_n; B)) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \int_B e^{-\frac12 <Q^{-1}x, x>} dx$

$B \in \mathbb{R}^n$

Proposition 1.2.1 *$\mathbb{P}$ has a unique extension to a probability measure on $(\Omega, \mathcal{F})$, still denoted by $\mathbb{P}$.*

$\mathbb{P}$ is called the *Wiener measure* on $[0, T]$.

Let $0 \le t_1 < \dots < t_n \le T$, and let γ be the mapping defined by

$$\gamma : \Omega \to \mathbb{R}^n,\ \omega \mapsto (\omega(t_1), \dots, \omega(t_n)).$$

By (1.2.1) it follows that $\boxed{\gamma \circ \mathbb{P} = \mathcal{N}(0, Q_{t_1,\dots,t_n})}$ where

$$< Q^{-1}_{t_1,\dots,t_n}\xi, \xi > = \frac{\xi_1^2}{t_1} + \frac{(\xi_2-\xi_1)^2}{t_2-t_1} + \cdots + \frac{(\xi_n-\xi_{n-1})^2}{t_n-t_{n-1}},\ \xi \in \mathbb{R}^n. \quad (1.2.2)\ (1)$$

The following result follows from Theorem 1.1.3.

Theorem 1.2.2 *Let $n \in \mathbb{N}$, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a bounded Borel measurable function. Then we have*

$\varphi(\gamma(\omega))$ $\qquad \gamma \circ \mathbb{P}(dy) = \mathbb{P}(\gamma^{-1}(dy))$

$$\int_\Omega \varphi(\omega(t_1), \dots, \omega(t_n))\mathbb{P}(d\omega) = \int_{\mathbb{R}^n} \varphi(y)\mathcal{N}(0, Q_{t_1,\dots,t_n})(dy)$$

$$= [(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})]^{-1/2} \quad (1.2.3)$$

$$\times \int_{\mathbb{R}^n} \varphi(\xi) e^{-\frac12\left[\frac{\xi_1^2}{t_1}+\cdots+\frac{(\xi_n-\xi_{n-1})^2}{t_n-t_{n-1}}\right]} d\xi_1\cdots d\xi_n.$$

Remark 1.2.3 Formula (1.2.3) clearly holds also for Borel measurable functions φ having polynomial growth.

The function $\Omega \to \mathbb{R} : \omega \mapsto \varphi(\omega(t_1), \dots, \omega(t_n))$ is said to be *cylindrical.*

Example 1.2.4 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. By (1.2.3) we have, for all $0 < t \le T$, and all $\xi \in \mathbb{R}$

$$\int_\Omega \varphi(\xi + \omega(t))\,\mathbb{P}(d\omega) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} \varphi(x+\xi)dx. \quad (1.2.4)$$

$\varphi * g_t$

Note that the function u defined by

$$u(t, \xi) = \int_\Omega \varphi(\xi + \omega(t))\,\mathbb{P}(d\omega)$$

$$\int_\Omega (\omega(t)-\omega(s))^m \mathbb{P}(d\omega) = \int_{\mathbb{R}} dx\,dy\, \frac{e^{-\frac{x^2}{2s}-\frac{(y-x)^2}{2(t-s)}}}{\sqrt{2\pi s}\sqrt{2\pi(t-s)}}(y-x)^m$$

is the classical solution to heat equation (in quanto essa modellizza la distribuzione delle temper. u nel dominio Ω) $u(t,\xi): \Omega\times[0,+\infty[\to\mathbb{R}$

$$\begin{cases} u_t(t,\xi) = \dfrac{1}{2}\, u_{\xi\xi}(t,\xi),\ t>0,\ \xi\in\mathbb{R} \\[2ex] u(0,\xi) = \varphi(\xi),\ \xi\in\mathbb{R}. \end{cases}$$

Example 1.2.5 By (1.2.3) we have, for all $0 \le s < t \le T$, and all $m\in\mathbb{N}$, $(t<s)$

$$\int_\Omega (\omega(t)-\omega(s))^m\, \mathbb{P}(d\omega) = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2(t-s)}} x^m\, dx$$

$$= \begin{cases} \dfrac{(m)!}{(m/2)!\,2^{m/2}}\,(t-s)^{m/2} \text{ if } m \text{ is even.} \\[2ex] 0 \text{ if } m \text{ is odd.} \end{cases} \quad (1.2.5)$$

$$\int_\Omega |\omega(t)-\omega(s)|^{2m}\,\mathbb{P}(d\omega) = \frac{1}{\sqrt{(2\pi)^2 t(s-t)}}\int_\Omega e^{-\frac{x_1^2}{2t}-\frac{(x_2-x_1)^2}{2(s-t)}}(x_1-x_2)^{2m}\,dx_1dx_2$$

– In particolare

① $\int_\Omega |\omega(t)-\omega(s)|^2\mathbb{P}(d\omega) = (s-t)$

② $\int_\Omega |\omega(t)-\omega(s)|^4\mathbb{P}(d\omega) = 3(s-t)^2$

We now consider a linear transformation that reduces $Q_{t_1,\dots,t_n}$ to a diagonal form.

Proposition 1.2.6 *Let $\{0\le t_1<\dots<t_n\le T\}$, and let ρ be the mapping*

$$\rho:\Omega\to\mathbb{R}^n,\ \omega\mapsto(\omega(t_1),\omega(t_2)-\omega(t_1),\dots,\omega(t_n)-\omega(t_{n-1})).$$

Then $\boxed{\rho\circ\mathbb{P} = \mathcal{N}(0,\operatorname{diag}\{t_1,t_2-t_1,\dots,t_n-t_{n-1}\}).}$ $= \mathbb{P}(\rho^{-1}(\cdot))$

Proof — We set here $Q = Q_{t_1,\dots,t_n}$ for simplicity. Let $T\in\mathcal{L}(\mathbb{R}^n)$ be defined by

$$T(\xi) = (\xi_1,\ \xi_2-\xi_1,\ \dots,\xi_n-\xi_{n-1}),\ \xi\in\mathbb{R}^n.$$

$T = I_n - R_n$, $I_n = \gamma$ di prop. 1.2.1, $R_n(\xi) = (0,\xi_1,\dots,\xi_{n-1})$ right shift.

Then we have from (1.2.2)

$$< Q^{-1}T^{-1}\eta, T^{-1}\eta >= \frac{\eta_1^2}{t_1}+\frac{\eta_2^2}{t_2-t_1}+\dots+\frac{\eta_n^2}{t_n-t_{n-1}},\ \eta\in\mathbb{R}^n.$$

It follows

$$(T^{-1})^*Q^{-1}T^{-1} = \operatorname{diag}\left\{\frac{1}{t_1},\frac{1}{t_2-t_1},\dots,\frac{1}{t_n-t_{n-1}}\right\}.$$

$$Q^{-1} = T^*\operatorname{diag}\left\{\frac{1}{t_1},\dots,\frac{1}{t_n-t_{n-1}}\right\}T$$

Note: $E[\omega(t_i)] = \int_\Omega(\omega(t_i)-\omega(t_{i-1})+\omega(t_{i-1}))\,\mathbb{P}(d\omega) = \omega(t_{i-1})$

$Var(\omega(t_i)) = t_i - t_{i-1}$

(1) B è la variabile aleatoria su $C_0([0,T])$ distribuita secondo la misura di Wiener $\mathbb{P}$

and so

$$Q = T^{-1} \operatorname{diag} \{t_1, t_2 - t_1, \ldots, t_n - t_{n-1}\}(T^*)^{-1},$$

which yields

$$TQT^* = \operatorname{diag} \{t_1, t_2 - t_1, \ldots, t_n - t_{n-1}\},$$

and the conclusion follows. ■

1.2.1 Brownian motion

For any $t \in [0,T]$ we define a function $B(t) : \Omega \to \mathbb{R}$ by setting

$$B(t)(\omega) = \omega(t), \ \omega \in \Omega. \tag{1.2.6}$$

B is called the *standard Brownian motion* on $[0,T]$. (1)

Come si comportano le funzioni ω fra t_{i-1} e t_i è indipendente dalla storia passata. (moto browniano delle particelle in sospensione)

Remark 1.2.7 From Proposition 1.2.6 it follows that if $0 \le t_1 < \ldots < t_n \le T$, functions

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}),$$

are independent. (3)

– Perché $f = (B(t_1), \ldots)$, $f \circ \mathbb{P} = \prod_{i=1}^{n}$

We end this subsection by stating a result that will play a key role in the sequel. The proof is straightforward and it is left to the reader.

Proposition 1.2.8 *Let $\{0 < t_1 < \ldots < t_n < \sigma < s \le T\}$, and let*

$$\varphi : \mathbb{R}^n \to \mathbb{R}, \ \psi : \mathbb{R} \to \mathbb{R}$$

be bounded and Borel measurable. Then we have

$$\int_\Omega \varphi(\omega(t_1), \cdots, \omega(t_n))\psi(\omega(s) - \omega(\sigma))\mathbb{P}(d\omega)$$

$$= \int_\Omega \varphi(\omega(t_1), \cdots, \omega(t_n))\mathbb{P}(d\omega) \int_\Omega \psi(\omega(s) - \omega(\sigma))\mathbb{P}(d\omega). \tag{1.2.7}$$

Basta vedere $\varphi(\omega(t_1), \ldots, \omega(t_n))$ come $\varphi = F(\omega(t_1), \omega(t_2) - \omega(t_1), \ldots, \omega(t_n) - \omega(t_{n-1}), \omega(\sigma) - \omega(t_n), \omega(s) - \omega(\sigma))$

Remark 1.2.9 Proposition 1.2.8 can be expressed by saying that if $0 \le t_1 < \ldots < t_n \le T$, then functions

$$(\omega(t_1), \cdots, \omega(t_n)) \text{ and } \omega(s) - \omega(\sigma)$$

are independent.

allora

$\int \varphi(\omega(t_1), \ldots, \omega(t_n)) \psi(\omega(s) - \omega(\sigma)) \mathbb{P}(d\omega) =$

$\int \varphi(y)\psi(y) \ \gamma \circ \mathbb{P}(dy) =$

$= \int \varphi(y) \ N(0, Q_{t_1, \ldots})(dy) \int \psi(z) \ N(0, t-s)(dy)$

---

[3] Functions $F_1, \ldots, F_n : \Omega \to \mathbb{R}^k$, $k \in \mathbb{N}$, are said to be *independent* if $(F_1, \ldots, F_n) \circ \mathbb{P} = \prod_{i=1}^{n} F_i \circ \mathbb{P}$. See e.g. Billingsly[2].

1.2.2 Probability of the set $BV(0,T)$

Let us give some additional notation. We denote by Σ the set of all decompositions

$$\sigma = \{0 = t_0 < t_1 < \ldots < t_n = T\}$$

of the interval $[0,T]$. For any $\sigma \in \Sigma$ we set

$$\boxed{|\sigma| = \max_{k=1,\ldots,n} (t_k - t_{k-1})}$$

The set Σ is endowed with the usual partial ordering

$$\boxed{\sigma_1 < \sigma_2 \iff |\sigma_1| < |\sigma_2|.}$$

For any $\omega \in \Omega$ we set

$$\boxed{I_1(\sigma)(\omega) = \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|.}$$

$$\boxed{I_2(\sigma)(\omega) = \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|^2.}$$

We define the *total variation* $V_1(\omega)$ of ω as

$$V_1(\omega) = \sup_{\sigma \in \Sigma} I_1(\sigma)(\omega).$$

It is easy to see that accade poiché se $\sigma_1 > \sigma_2 \Rightarrow I_1(\sigma_1) \le I_1(\sigma_2)$

$$\lim_{|\sigma| \to 0} I_1(\sigma)(\omega) = V_1(\omega).$$

We shall denote by $\boxed{BV(0,T)}$ the set of all functions $\omega \in \Omega$ having finite total variation.

We want to show that $\mathbb{P}(BV(0,T)) = 0$. To this purpose we introduce the *quadratic variation* $\boxed{V_2(\omega)}$ of ω, defined by

$$\boxed{V_2(\omega) = \lim_{|\sigma| \to 0} I_2(\sigma)(\omega).}$$

when the limit exists. Notice that if $\boxed{V_1(\omega)}$ is finite, we have $\boxed{V_2(\omega) = 0.}$ This follows from the inequality

$$\boxed{I_2(\sigma)(\omega) \le \sup_{i=1,2,\ldots,n} |\omega(t_i) - \omega(t_{i-1})| V_1(\omega)}$$ fissato σ

per ciò passare al $\lim |\sigma| \downarrow 0 \Rightarrow \sup_{i=1,2,\ldots,n} |\omega(t_i) - \omega(t_{i-1})| \to 0$

For any $\omega \in \Omega$ we set

$$\boxed{I_1(\sigma)(\omega) = \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|,}$$

$$\boxed{I_2(\sigma)(\omega) = \sum_{i=1}^{n} |\omega(t_i) - \omega(t_{i-1})|^2.}$$

We define the *total variation* $V_1(\omega)$ of ω as

$$\boxed{V_1(\omega) = \sup_{\sigma \in \Sigma} I_1(\sigma)(\omega).}$$

It is easy to see that

$$\boxed{\lim_{|\sigma| \to 0} I_1(\sigma)(\omega) = V_1(\omega).}$$

We shall denote by $\boxed{BV(0,T)}$ the set of all functions $\omega \in \Omega$ having finite total variation.

We want to show that $\mathbb{P}(BV(0,T)) = 0$. To this purpose we introduce the *quadratic variation* $V_2(\omega)$ of ω, defined by

$$\boxed{V_2(\omega) = \lim_{|\sigma| \to 0} I_2(\sigma)(\omega),}$$

when the limit exists. Notice that if $V_1(\omega)$ is finite, we have $V_2(\omega) = 0$. This follows from the inequality

$$I_2(\sigma)(\omega) \le \sup_{i=1,2\dots,n} |\omega(t_i) - \omega(t_{i-1})| V_1(\omega).$$

So, to prove that $V_1(\omega) = +\infty$ it is enough to show that $V_2(\omega) > 0$.

To prove that $\boxed{\mathbb{P}(BV(0,T)) = 0}$ we need the following result.

Proposition 1.2.11 *We have*

$$\lim_{|\sigma| \to 0} \int_\Omega |I_2(\sigma)(\omega) - T|^2 \, \mathbb{P}(d\omega) = 0.$$

Proof — We first remark that by (1.2.5) we have

$$\begin{aligned} \int_\Omega I_2(\sigma)(\omega) \, \mathbb{P}(d\omega) &= \sum_{i=1}^{n} \int_\Omega |\omega(t_i) - \omega(t_{i-1})|^2 \, \mathbb{P}(d\omega) \\ &= \sum_{i=1}^{n} (t_i - t_{i-1}) = T. \end{aligned}$$

We have moreover

$$\int_\Omega [I_2(\sigma)(\omega)]^2\,\mathbb{P}(d\omega) = \sum_{i=1}^n \int_\Omega |\omega(t_i)-\omega(t_{i-1})|^4\,\mathbb{P}(d\omega)$$

$$+2\sum_{i<j}\int_\Omega |\omega(t_i)-\omega(t_{i-1})|^2|\omega(t_j)-\omega(t_{j-1})|^2\,\mathbb{P}(d\omega).$$

By (1.2.5) it follows that

$$\int_\Omega |\omega(t_i)-\omega(t_{i-1})|^4\,\mathbb{P}(d\omega) = 3(t_i-t_{i-1})^2. \tag{1.2.8}$$

Moreover, by (1.2.7) if $i<j$

$$\begin{aligned}&\int_\Omega |\omega(t_i)-\omega(t_{i-1})|^2|(\omega(t_j)-\omega(t_{j-1})|^2\,\mathbb{P}(d\omega)\\ &= \int_\Omega |\omega(t_i)-\omega(t_{i-1})|^2\,\mathbb{P}(d\omega)\int_\Omega |\omega(t_j)-\omega(t_{j-1})|^2\,\mathbb{P}(d\omega)\\ &= (t_i-t_{i-1})(t_j-t_{j-1}).\end{aligned} \tag{1.2.9}$$

By (1.2.8) and (1.2.9) it follows

$$\int_\Omega [I_2(\sigma)(\omega)]^2\,\mathbb{P}(d\omega) = 2\sum_{i=1}^n (t_i-t_{i-1})^2 + \left(\sum_{i=1}^n (t_i-t_{i-1})\right)^2$$

$$= 2\sum_{i=1}^n (t_i-t_{i-1})^2 + T^2.$$

Consequently

$$\begin{aligned}\int_\Omega |I_2(\sigma)(\omega)-T|^2\,\mathbb{P}(d\omega) &= \int_\Omega [I_2(\sigma)(\omega)]^2\,\mathbb{P}(d\omega) - T^2\\ &\le 2|\sigma|T \to 0 \text{ for } |\sigma|\to 0.\end{aligned}$$

∎

Theorem 1.2.12 *$BV(0,T)$ is a Borel set and*

$$\mathbb{P}(BV(0,T)) = 0. \tag{1.2.10}$$

(1) quindi dalla Prop. 1.2.11 $\mathbb{E}[|I_2(\sigma_n) - T|] \to 0 \Rightarrow \mathbb{P}[|I_2(\sigma_n) - T|] \to 0$
$\Rightarrow \exists \{\sigma_{n_k}\} : |I_2(\sigma_{n_k}) - T| \to 0$ $\mathbb{P}$ q.c. allora
$\Gamma \in \mathcal{F}$: $\forall \omega \in \Gamma$ $V_2(\omega) = T \Rightarrow \mathbb{P}(\Gamma) = 1 \Rightarrow BV(0,T) \subset \Gamma^c$

Proof — Denote by $\Sigma_{\mathbb{Q}}$ the subset of Σ of all decompositions

$$\sigma = \{0 = t_0 < t_1 < ... < t_n = T\}$$

such that $t_i \in \mathbb{Q}$, $i = 1, ..., n-1$. Since

$I_2(\sigma)(\omega)$ è una funz. mis
$I_2(\sigma) : \Omega \to \mathbb{R}$

$$* \quad V_1(\omega) = \lim_{|\sigma| \to 0, \sigma \in \Sigma_{\mathbb{Q}}} I_1(\sigma)(\omega)$$

$BV(0,T)$ is a Borel set. $\Omega = \mathcal{C}_0([0,T])$.

We note now that from Proposition 1.2.11 there exists a sequence $\{\sigma_n\} \subset \Sigma$ with $\lim_{n\to\infty} |\sigma_n| = 0$, and a Borel set $\Gamma \subset \mathcal{F}$ such that $\mathbb{P}(\Gamma) = 1$ and [3]

$$V_2(\omega) = \lim_{n\to\infty} I_2(\sigma_n)(\omega) = T, \ \forall\, \omega \in \Gamma.$$ $\Rightarrow$ per la (1.2.X bis) che $V_1(\omega) = +\infty$ $\forall \omega \in \Gamma$

Thus $BV(0,T) \subset \Gamma^c$, and so (1.2.10) holds.(1)

1.2.3 Probability of the set $C_0^\alpha([0,T])$

For any $\alpha \in]0,1[$ we set

$$\boxed{\|\omega\|_\alpha = \sup_{0 \le s < t \le T} \frac{|\omega(t) - \omega(s)|}{|t-s|^\alpha}, \ \forall\, \omega \in \Omega,}$$

and

$$\boxed{C_0^\alpha([0,T]) = \{\omega \in \Omega : \ \|\omega\|_\alpha < +\infty\}.}$$

$C_0^\alpha([0,T])$ is the set of all real α-Hölder continuous functions on $[0,T]$ vanishing at 0. It is a Banach space with the norm $\boxed{\|\cdot\|_\alpha}$. Note that since

$$\|\omega\|_\alpha = \sup_{t,s \in \mathbb{Q}, t \neq s} \frac{|\omega(t) - \omega(s)|}{|t-s|^\alpha},$$

the mapping

$$\phi : \Omega \to [0, +\infty], \ \omega \mapsto \|\omega\|_\alpha$$ perché sup di una succ. numerabile Borel-mis.

is Borel measurable, so that $\boxed{C_0^\alpha([0,T]) \in \mathcal{F}}$.

[3]Notice that the set Γ is somewhat strange. In fact if $\omega \in \Gamma$ and ω is not 0, then $\alpha\omega \notin \Gamma$ for any $\alpha \in \mathbb{R}$ different from 1 and -1.

Fissiamo una partizione $0 = t_0 < t_1 < \dots < t_n = T$ a valori in $\mathbb{Q}$ si ottiene che

$$|\omega(t_i)| \le \frac{|\omega(t_i) - \omega(t_0)|}{|t_i - t_0|^\alpha} \cdot |t_i - t_0|^\alpha \le \sup_{t,s \in \mathbb{Q}} \frac{|\omega(t) - \omega(s)|}{|t-s|^\alpha}$$

$$|\omega(t_i)| \le \|\omega\|_\alpha$$

Da cui $\omega \in I(t_1, \dots, t_n; [-\|\omega\|_\alpha, \|\omega\|_\alpha]^n) \Rightarrow \phi^{-1}(B(m,n)) = \bigcup$

Theorem 1.2.13 *Assume that $\alpha \in]1/2, 1[$.* [4] *Then*

$$\mathbb{P}(C_0^\alpha([0,T])) = 0. \tag{1.2.11}$$

Proof — Let $\omega \in C_0^\alpha([0,T])$ and $\sigma = \{0 = t_0 < t_1 < \ldots < t_n = T\} \in \Sigma$. Then we have

$$I_2(\sigma)(\omega) \;\le\; \|\omega\|_\alpha^2 \sum_{j=1}^{n} |t_i - t_j|^{2\alpha} \le \; \|\omega\|_\alpha^2 \max_{i,j=1,\ldots,n} |t_i - t_j|^{2\alpha-1} T.$$

$V_2(\omega) = T$ $\mathbb{P}$-q.o ω Since $2\alpha > 1$ it follows that $V_2(\omega) = 0$. This implies

$$C_0^\alpha([0,T]) \subset \Gamma^c,$$

where Γ is the set defined in the proof of Theorem 1.2.12. Since $\mathbb{P}(\Gamma) = 1$ the conclusion follows. ∎

We want now to show that when $\boxed{\alpha \in]0, 1/2[}$ we have $\boxed{\mathbb{P}(C_0^\alpha([0,T])) = 1.}$ For this it is useful to introduce fractional Sobolev spaces. (famiglia di spazi intermedi fra L^p e $W^{1,p}$)

For any $\boxed{p \ge 1}$ and any $\boxed{\beta \in]0, 1[}$ we set

$$\boxed{\|\omega\|_{W_0^{\beta,p}(0,T)} = \left(\int_0^T \int_0^T \frac{|\omega(t) - \omega(s)|^p}{|t-s|^{1+\beta p}} dt\, ds \right)^{1/p}, \quad \omega \in \Omega,}$$

and

$$\boxed{W_0^{\beta,p}(0,T) = \{\omega \in \Omega : \; \|\omega\|_{W_0^{\beta,p}(0,T)} < +\infty\}.}$$

$W_0^{\beta,p}$ is a Banach space with the norm $\|\cdot\|_{W_0^{\beta,p}(0,T)}$. We recall the Sobolev embedding theorem [5]

Proposition 1.2.14 *Let $p \ge 1$ and $\beta \in]0, 1]$. If $\beta > \frac{1}{p}$ then*

$$W_0^{\beta,p}(0.T) \subset C_0^{\beta - \frac{1}{p}}([0,T]),$$

and there exists $C_{\beta,p} > 0$ such that

$$\|\omega\|_{\beta - \frac{1}{p}} \le C_{\beta,p} \, \|\omega\|_{W_0^{\beta,p}(0,T)}, \; \forall\, \omega \in \Omega. \tag{1.2.12}$$

[4] The conclusion of theorem holds also for $\boxed{\alpha = 1/2.}$ See e. g. D. Revuz and M. Yor [5], page 29.

[5] See e.g. R. A. Adams [1]

Teorema (Sobolev): $m \ge 1$ $(m \in \mathbb{N})$, $1 \le p \le \infty$

① $1/p - \frac{m}{N} > 0$ $\quad W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ $\quad \frac{1}{q} = \frac{1}{p} - \frac{m}{N}$

② $\frac{1}{p} - \frac{m}{N} = 0$ $\quad W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ $\quad \forall q \in [p, +\infty[$

③ $\frac{1}{p} - \frac{m}{N} < 0$ $\quad W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$

Chapter 2

Stochastic integral

In all this chapter $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space defined in Chapter 1. For any $p \in [1, +\infty[$ we denote by $\boxed{\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})}$ the set of all Borel mappings $X : \Omega \to \mathbb{R}$ such that

$$\boxed{\int_\Omega |X(\omega)|^p \, \mathbb{P}(d\omega) < +\infty.}$$

$\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ is a linear space with the usual linear operations. We shall denote by $L^p(\Omega, \mathcal{F}, \mathbb{P})$ the quotient space of $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the equivalence relation

$$X \sim Y \Longleftrightarrow \int_\Omega |X(\omega) - Y(\omega)| \, \mathbb{P}(d\omega) = 0.$$

If $X \sim Y$ we write $X = Y$, $\mathbb{P}$ – a.e.. As well known, $\boxed{L^p(\Omega, \mathcal{F}, \mathbb{P}),}$ endowed with the norm

$$\boxed{\|X\|_p = \left(\int_\Omega |X(\omega)|^p \, \mathbb{P}(d\omega)\right)^{1/p}.}$$

is a Banach space.

2.1 Wiener integral

We want here to define the integral

$$\boxed{I(f) = \int_0^T f(s) dB(s), \ \omega \in \Omega,}$$

where $f \in \mathcal{L}^2([0,T])$. ([1]) We will set

$$\boxed{I(f)(\omega) = \int_0^T f(s)d\omega(s),\ \omega \in \Omega.}$$

integrale di Stieltjes
se $\omega \in C^1$
$\int f d\omega := \int_0^T f \omega' dx$
se $\omega \in BV$
$\int f d\omega := \lim \sum f(t_{i-1})(\omega(t_i) - \omega(t_{i-1}))$

This expression can be interpreted as a Stiltjies integral when ω has bounded variation, but we know from Theorem 1.2.12 that this is almost never the case. So we will give a different meaning to the integral expression above, showing that suitable Riemann sums are convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We denote by $S(0,T)$ the set of all mappings $f : [0,T] \to \mathbb{R}$ of the form

$$\boxed{f = \sum_{k=1}^{n} f_{k-1} \chi_{[t_{k-1}, t_k[},}$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_n = T$, and $f_0, f_1, \dots, f_{n-1} \in \mathbb{R}$.

For any $f \in S(0,T)$ we set

$$\boxed{I(f)(\omega) = \sum_{k=1}^{n} f_{k-1}(\omega(t_k) - \omega(t_{k-1})).}$$

The following lemma is basic.

Lemma 2.1.1 *Let $f \in S(0,T)$. Then we have*

$$\boxed{\int_\Omega I(f)(\omega)\, \mathbb{P}(d\omega) = 0} \quad \text{(media nulla)} \tag{2.1.1}$$

and

$$\boxed{\int_\Omega |I(f)(\omega)|^2\, \mathbb{P}(d\omega) = \int_0^T |f(s)|^2 ds.} \quad (\text{Varianza} = \|f\|^2_{L^2}) \tag{2.1.2}$$

Proof — By (1.2.5) it follows

$$\int_\Omega \underset{\text{"}[\omega(t) - \omega(0)]\text{"}}{\omega(t)}\, \mathbb{P}(d\omega) = 0,\ \forall\, t \geq 0,$$

[1] $\mathcal{L}^2([0,T])$ is the set of all Borel mappings $f : [0,T]) \to \mathbb{R}$ such that

$$\boxed{\int_0^T |f(t)|^2\, dt < +\infty.}$$

so that (2.1.1) holds. Let us prove (2.1.2). We have

$$\int_\Omega |I(f)(\omega)|^2\,\mathbb{P}(d\omega) = \int_\Omega \sum_{j=1}^n |f(t_{j-1})|^2[\omega(t_j)-\omega(t_{j-1})]^2\,\mathbb{P}(d\omega)$$

$$+2\int_\Omega \sum_{j<k}^n f(t_{j-1})f(t_{k-1})[\omega(t_j)-\omega(t_{j-1})][\omega(t_k)-\omega(t_{k-1})]\,\mathbb{P}(d\omega). \tag{2.1.3}$$

By (1.2.5) we have

$$\int_\Omega [\omega(t_j)-\omega(t_{j-1})]^2\,\mathbb{P}(d\omega) = t_j - t_{j-1}.$$

Moreover, by Proposition ?? it follows

$$\int_\Omega [\omega(t_j)-\omega(t_{j-1})][\omega(t_k)-\omega(t_{k-1})]\,\mathbb{P}(d\omega) = 0.$$

Per l'indipendenza degli incrementi, è il prodotto di due integrali nulli.

By substituting in (2.1.3) we find finally

$$\int_\Omega |I(f)(\omega)|^2\mathbb{P}(d\omega) = \sum_{j=1}^n |f(t_{j-1})|^2(t_j - t_{j-1}). \blacksquare$$

(f è a scala)

Now, let $f \in \mathcal{L}^2(0,T)$. For any $\sigma \in \Sigma$, (²) $\sigma = \{0 = t_0 < t_1 < \ldots < t_n = T\}$, we consider the function $f_\sigma \in S(0,T)$ defined as

$$\boxed{f_\sigma = \sum_{j=1}^n f(t_{k-1})\chi_{[t_{k-1},t_k[}.} \tag{2.1.4}$$

Theorem 2.1.2 *For any $f \in \mathcal{L}^2([0,T])$ there exists the limit*

$$\boxed{\lim_{|\sigma|\to 0} I(f_\sigma) =: \int_0^T f(s)dB(s).} \tag{2.1.5}$$

in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Moreover we have

$$\boxed{\int_\Omega \left(\int_0^T f(s)d\omega(s)\right)\mathbb{P}(d\omega) = 0,} \tag{2.1.6}$$

and

$$\boxed{\int_\Omega \left|\int_0^T f(s)d\omega(s)\right|^2 \mathbb{P}(d\omega) = \int_0^T |f(s)|^2 ds.} \tag{2.1.7}$$

[2] the set of all decompositions of the interval $[0,T]$ defined in §1.2.1.

Proof— We prove that $\{I(f_\sigma)\}$ is a Cauchy sequence on $\boxed{L^2(\Omega,\mathcal{F},\mathbb{P})}$. In fact, by (2.1.2) it follows, for $\sigma,\eta\in\Sigma$,

$$\int_\Omega |I(f_\sigma)-I(f_\eta)|^2\mathbb{P}(d\omega)=\int_0^T |f_\sigma(t)-f_\eta(t)|^2dt.$$

Since obviously $\{f_\sigma\}$ is a Cauchy sequence on $L^2(0,T)$, the conclusion follows.

We have finally

$$\int_\Omega\left(\int_0^T f(s)d\omega(s)\right)\ \mathbb{P}(d\omega)=\lim_{|\sigma|\to 0}\int_\Omega I(f_\sigma)(\omega)\ \mathbb{P}(d\omega)=0,$$

and

$$\int_\Omega\left|\int_0^T f(s)d\omega(s)\right|^2\ \mathbb{P}(d\omega)=\lim_{|\sigma|\to 0}\int_\Omega |I(f_\sigma)(\omega)|^2\ \mathbb{P}(d\omega)$$

$$=\lim_{|\sigma|\to 0}\sum_{j=1}^n |f(t_{j-1})|^2(t_j-t_{j-1})=\int_0^T |f(s)|^2ds.\ \blacksquare$$

The function of $L^2(\Omega,\mathcal{F},\mathbb{P})$

$$\boxed{I(f)=\int_0^T f(s)dB(s),\ \omega\in\Omega.}$$

is called the *Wiener integral* of f in $[0,T]$.

We define in obvious way the Wiener integral $\int_a^b f(s)dB(s)$ in any interval $[a,b]\subset[0,T]$.

We note that if $f\in C^1([0,T])$ then it is possible to express the Wiener integral $\int_0^T f(s)dB(s)$ in terms of a Riemann integral as the following integration by parts formula shows.

Proposition 2.1.3 *If $f\in C^1([0,T])$ we have*

$$\boxed{\int_0^T f(s)dB(s)=f(T)B(T)-\int_0^T f'(s)B(s)ds.}\tag{2.1.8}$$

Proof — Let $\sigma = \{0 = t_0 < t_1 < ... < t_n = T\} \in \Sigma$. Then we have

$$\begin{aligned} I(f_\sigma)(\omega) &= \sum_{k=1}^{n} f(t_{k-1})(\omega(t_k) - \omega(t_{k-1})) \\ &= \sum_{k=1}^{n}(f(t_k)\omega(t_k) - f(t_{k-1})\omega(t_{k-1})) \\ &- \sum_{k=1}^{n}(f(t_k) - f(t_{k-1}))\omega(t_k) \\ &= f(T)\omega(T) - \sum_{k=1}^{n}(f(t_k) - f(t_{k-1}))\omega(t_k) \\ &= f(T)\omega(T) - \sum_{k=1}^{n} f'(\alpha_k)\omega(t_k)(t_k - t_{k-1}), \end{aligned}$$

where α_k is a suitable number in the interval $[t_{k-1}, t_k], k = 1, ..., n$. It follows that

$$\lim_{|\sigma|\to 0} I(f_\sigma)(\omega) = f(T)\omega(T) - \int_0^T f'(s)\omega(s)ds, \quad \mathbb{P} - \text{a.e. } \omega \in \Omega. \ \blacksquare$$

2.2 Itô integral

In this section we set for simplicity $L^2(\Omega, \mathcal{F}. \mathbb{P}) = L^2(\Omega)$. We want to define a stochastic integral

$$\boxed{\int_0^T F(s)(\omega)d\omega(s),}$$

for a class of functions ([3])

$$[0, T] \to L^2(\Omega), \ t \mapsto F(t),$$

[3]A *stochastic process* on $[0, T]$ is a function $F : [0, T] \to L^2(\Omega)$. A function $F \in C([0, T]; L^2(\Omega))$ is called a stochastic process *continuous in mean square* .

belonging to $C([0,T];L^2(\Omega))$. By $C([0,T];L^2(\Omega))$ we mean the Banach space of all continuous mappings from $[0,T]$ into $L^2(\Omega)$, endowed with the norm

$$\|F\|_{C([0,T];L^2(\Omega))} = \sup_{t\in[0,T]} \left(\int_\Omega |F(t)(\omega)|^2 \, \mathbb{P}(d\omega) \right)^{1/2}.$$

We will define the above integral for those mappings F belonging to $C([0,T];L^2(\Omega))$ such that $F(t)$, $t \in [0,T]$, depends in some sense only on the values of ω on $[0,t]$.

We set $\mathcal{F}_0 = \{\Omega, \emptyset\}$, and for any $t \in]0,T]$ we denote by $\mathcal{F}_t$ the smallest σ -algebra in $\mathcal{F}$ containing all cylindrical sets of the form

$$I(t_1,...,t_n;B) = \{\omega \in \Omega : (\omega(t_1),...,\omega(t_n)) \in B\},$$

where $n \in \mathbb{N}$, $0 < t_1 < ... < t_n \le t$ and $B \in \mathcal{B}(\mathbb{R}^n)$.

The family of σ–algebras $\{\mathcal{F}_t\}_{t\in[0,T]}$ is called the *natural filtration* of $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 2.2.1 *Let X be a cylindrical function $\mathcal{F}_t$-measurable. Then there exist $n \in \mathbb{N}$, $0 < t_1 < ... < t_n \le t$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ Borel measurable, such that*

$$X(\omega) = \varphi(\omega(t_1),...,\omega(t_n)),\ \omega \in \Omega. \tag{2.2.1}$$

$$\begin{tikzcd} \Omega \arrow[r] \arrow[dr, "X"'] & \mathbb{R}^n \arrow[d, "\varphi"] \\ & \mathbb{R} \end{tikzcd}$$
$\omega \longmapsto (\omega(t_1), ..., \omega(t_n))$

Proof— By the hypothesis there exists a minimal positive integer n, positive numbers $t_1 < ... < t_n \le T$, and a Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$, such that (2.2.1) holds.

If φ is constant the result is obviously true. Assume that φ is not constant, and let $B \in \mathcal{B}(\mathbb{R})$ be such that $\varphi^{-1}(B) \ne \mathbb{R}$. Then the set

$$\{\omega \in \Omega : \varphi(\omega(t_1),...,\omega(t_n)) \in B\} = \left\{\omega \in \Omega : (\omega(t_1),...,\omega(t_n)) \in \varphi^{-1}(B)\right\}$$

Since X is $\mathcal{F}_t$–measurable we have

$$\left\{\omega \in \Omega : (\omega(t_1),...,\omega(t_n)) \in \varphi^{-1}(B)\right\} \in \mathcal{F}_t,$$

that yields $t_n \le t$ in view of the minimality of n. ∎

ω è prevedibile se misurabile rispetto a $\mathcal{F}_t$ (...

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Remark 2.2.2 $\mathcal{L}^p(\Omega, \mathcal{F}_0, \mathbb{P})$ and $L^p(\Omega, \mathcal{F}_0, \mathbb{P})$ are both isomorphic to $\mathbb{R}$.

Obviously we have

$$\mathcal{L}^p(\Omega, \mathcal{F}_t, \mathbb{P}) \subset \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}), \ \forall\, t \in [0, T].$$

For all $t \in [0, T]$, there is a natural identification of $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ with a closed subspace of $L^p(\Omega, \mathcal{F}, \mathbb{P})$, still denoted $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$.

Let $F \in C([0, T]; L^2(\Omega))$. If $F(t)$ belongs to $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $t \in [0, T]$ then F is said to be *adapted*. We shall denote by $\boxed{C_{ad}([0, T])}$ the set of all functions in $C([0, T]; L^2(\Omega))$ that are adapted. As easily checked $C_{ad}([0, T])$ is a closed subspace of $C([0, T]; L^2(\Omega))$, and it will be endowed with its norm.

The concept of adapted function is of basic importance in the definition of Itô integral below.

Exercise 2.2.3 Prove the following statements .

(i) B belongs to $C([0, T]; L^2(\Omega))$ and it is adapted.

(ii) If $f \in C([0, T])$ then the function $F : t \mapsto F(t)$, where

$$F(t)(\omega) = \int_0^t f(s)dB(s),$$

is adapted.

(iii) If $f \in C([0, T])$ is not identically equal to 0, then the function $F : t \mapsto F(t)$, where

$$F(t)(\omega) = \int_t^T f(s)dB(s),$$

is not adapted.

The following result is an easy consequence of Proposition 1.2.9. It is of key importance in the construction of the Itô integral.

Proposition 2.2.4 *Let $X \in L^2(\Omega, \mathcal{F}_s, \mathbb{P})$, $0 \leq s < t \leq T$. If $\psi : \mathbb{R} \to \mathbb{R}$ is Borel measurable and bounded, we have*

$$\int_\Omega X(\omega)\psi(\omega(t) - \omega(s))\, \mathbb{P}(d\omega) = \int_\Omega X(\omega)\, \mathbb{P}(d\omega) \int_\Omega \psi(\omega(t) - \omega(s))\, \mathbb{P}(d\omega). \tag{2.2.2}$$

Perché $\int_t^T f(s)dB(s) = \int_0^T f(s)dB(s) - \int_0^t f(s)dB(s)$

2\. $F \in \mathcal{C}([0,T], L^2(\Omega))$

$F(t) \in L^2(\Omega)$: $\int_\Omega |\int_0^t f(s)dB(s)|^2 = \int_0^t f(s)^2 ds$

$F(t)$ è limite puntuale di funzioni (num.) $\mathcal{F}_t$-mis.

1\. $B \in \mathcal{C}([0,T], L^2(\Omega))$ e adattato

- $B : [0,T] \longrightarrow L^2(\Omega)$ continua perché $\|B(t) - B(s)\|^2_{L^2(\Omega)} = \int_\Omega |(B(t) - B(s))(\omega)|^2 P(d\omega) = \int_\Omega |\omega(t) - \omega(s)|^2 = t - s$

- $\sup_{t \in [0,T]} \left(\int_\Omega B(t)(\omega)^2 P(d\omega)\right)^{1/2} = \sup_{t \in [0,T]} \left(\int_\Omega (\omega(t) - \omega(0))^2 P(d\omega)\right)^{1/2} = \sqrt{T}$

- $B(t) \in L^2(\Omega, \mathcal{F}_t, P)$ $\forall t$ perché $B(t)^{-1}(B(0,r)) = \{\omega \mid \omega(t) \in B\} = \bigcup \ldots$

1. $\psi_n(\omega) = \varphi_n(\omega(t_1),\dots,\omega(t_n)) \to \psi(\omega) = \varphi(\omega(t_1),\dots)$ (φ_n di Borel puntualmente limitate ... e limitate)

Allora $\psi \in \mathcal{Y}$

2. D contiene le costanti e le funz caratteristiche dei cilind.
$\chi_{I(t_1,\dots,t_n|B)} = \chi(\omega(t_1),\dots,\omega(t_n))$

Proof— It is enough to prove the result when X is cylindrical,

$$X(\omega) = \varphi(\omega(\alpha_1),\dots,\omega(\alpha_n)),$$

da 2.2.1 with $n \in \mathbb{N}$, $0 < \alpha_1 < \dots < \alpha_n \le s$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ bounded and Borel measurable. In this case the conclusion follows from Proposition 1.2.9. ∎

Proposition 2.2.4 can be expressed by saying that if X is $\mathcal{F}_s$ measurable and $\boxed{s < t}$ then X and $B(t) - B(s)$ are independent.

2.2.1 Future filtration

For any $s \ge 0$ we denote by $\mathcal{F}^s$ the sub–σ–algebra of $\mathcal{F}$ generated by all sets of the form

$$\boxed{\{\omega \in \Omega : (\omega(t_1) - \omega(s),\dots,\omega(t_n) - \omega(s)) \in B\}},$$

where $n \in \mathbb{N}$, $s \le t_1 \dots < t_n \le T$ and $B \in \mathcal{B}(\mathbb{R}^n)$. The family $\{\mathcal{F}^s\}_{s\in[0,T]}$ is called the *future filtration*.

The following result can be proved as Proposition 2.2.4.

Proposition 2.2.5 *Let $X \in L^2(\Omega, \mathcal{F}^s, \mathbb{P})$, $0 \le t < s \le T$. If $\psi : \mathbb{R} \to \mathbb{R}$ is Borel measurable and bounded, we have*

$$\int_\Omega X(\omega)\psi(\omega(t) - \omega(s))\,\mathbb{P}(d\omega) = \int_\Omega X(\omega)\,\mathbb{P}(d\omega)\int_\Omega \psi(\omega(t) - \omega(s))\,\mathbb{P}(d\omega). \tag{2.2.3}$$

Quindi se $|t_{k-1} - t| < |t_{k-1} - t_k| < \delta(\varepsilon)$ allora $\int |F(t_{k-1})(\omega) - F(t)(\omega)|^2 \mathbb{P}(d\omega) < \varepsilon$

Proposition 2.2.5 can be expressed by saying that if X is $\mathcal{F}^s$ measurable and $t < s$, then X and $B(s) - B(t)$ are independent.

cui se $|\sigma| \to 0$, definitivamente $|\sigma| = \sup|t_k - t_{k-1}| < \delta(\varepsilon)$ cioè $|t_{k-1} - t_k| < \delta(\varepsilon)$ $\forall k$.

2.2.2 Definition of Itô integral

Let $F \in C_{ad}([0,T])$. For any $\sigma = \{0 = t_0 < t_1 < \dots < t_n = T\} \in \Sigma$ we consider the integral sum

Quindi $\forall t$ vale che $\int_\Omega |F_\sigma(t)(\omega) - F(t)(\omega)|^2 \mathbb{P}(d\omega) < \varepsilon$ cioè (*).

$$\boxed{I_\sigma(F) = \sum_{j=1}^n F(t_{j-1})(B(t_j) - B(t_{j-1})).} \tag{2.2.4}$$

- $F_\sigma = \sum_{k=1}^n F(t_{k-1})\,\chi_{[t_{k-1},t_k]}$

$F_\sigma \xrightarrow[|\sigma|\to 0]{} F$ in $\mathcal{C}([0,T], L^2(\Omega))$

(*) $\sup_{t\in[0,T]} \left(\int_\Omega |F_\sigma(t)(\omega) - F(t)(\omega)|^2 \mathbb{P}(d\omega)\right)^{1/2} \xrightarrow[|\sigma|\to 0]{} 0$

in quanto: $F \in \mathcal{C}^0([0,T], \dots)$, in particolare F unif. $\mathcal{C}^0$ su $[0,T]$. Allora preso $t \in \dots$

$\int_\Omega |F_\sigma(t)(\omega) - F(t)(\omega)|^2 = \int_\Omega |F(t_{k-1})(\omega) - F(t)(\omega)|^2$

Our goal is to prove that there exists the limit

$$\boxed{\lim_{|\sigma|\to 0} I_\sigma(F) =: \int_0^T F(s)dB(s), \text{ in } L^2(\Omega,\mathcal{F},\mathbb{P}).}$$

As in the definition of Wiener integral we prove first a lemma.

Lemma 2.2.6 *Let $F \in C_{ad}([0,T])$ and let $\sigma \in \Sigma$. Then we have*

$$\int_\Omega I_\sigma(F)(\omega)\,\mathbb{P}(d\omega) = 0 \tag{2.2.5}$$

and

$$\int_\Omega |I_\sigma(F)(\omega)|^2\,\mathbb{P}(d\omega) = \int_\Omega \sum_{j=1}^n |F(t_{j-1})(\omega)|^2(t_j - t_{j-1})\,\mathbb{P}(d\omega). \tag{2.2.6}$$

Proof — Let us prove (2.2.5). We have

$$\int_\Omega I_\sigma(F)(\omega)\,\mathbb{P}(d\omega) = \sum_{j=1}^n \int_\Omega F(t_{j-1})(\omega)(\omega(t_j) - \omega(t_{j-1}))\,\mathbb{P}(d\omega).$$

Since $F(t_{j-1}) \in L^2(\Omega, \mathcal{F}_{t_{j-1}}, \mathbb{P})$, from Proposition 2.2.4 it follows

$$\int_\Omega I_\sigma(F)(\omega)\,\mathbb{P}(d\omega)$$

$$= \sum_{j=1}^n \int_\Omega F(t_{j-1})(\omega)\,\mathbb{P}(d\omega) \int_\Omega (\omega(t_j) - \omega(t_{j-1}))\,\mathbb{P}(d\omega) = 0.$$

Let us prove finally (2.2.6). We have

$$\int_\Omega |I_\sigma(F)(\omega)|^2\,\mathbb{P}(d\omega) = \int_\Omega \sum_{j=1}^n |F(t_{j-1})(\omega)|^2[\omega(t_j) - \omega(t_{j-1})]^2\,\mathbb{P}(d\omega)$$

$$+2\int_\Omega \sum_{j<k}^n F(t_{j-1})(\omega)F(t_{k-1})(\omega)[\omega(t_j) - \omega(t_{j-1})]\,[\omega(t_k) - \omega(t_{k-1})]\mathbb{P}(d\omega).$$

Since $j < k$ the function

$$\Omega \to \mathbb{R},\ \omega \mapsto F(t_{j-1})(\omega)F(t_{k-1})(\omega)[\omega(t_j) - \omega(t_{j-1})],$$

is $\mathcal{F}_{t_{k-1}}$-measurable. Then applying again Proposition 2.2.4 we find

$$\int_\Omega F(t_{j-1})(\omega)F(t_{k-1})(\omega)[\omega(t_j)-\omega(t_{j-1})][\omega(t_k)-\omega(t_{k-1})]\;\mathbb{P}(d\omega)$$

$$=\int_\Omega F(t_{j-1})(\omega)F(t_{k-1})(\omega)[\omega(t_j)-\omega(t_{j-1})]\;\mathbb{P}(d\omega)$$

$$\times\int_\Omega[\omega(t_k)-\omega(t_{k-1})]\;\mathbb{P}(d\omega)=0.$$

It follows

$$\int_\Omega |I_\sigma(F)(\omega)|^2\;\mathbb{P}(d\omega)=\sum_{j=1}^n\int_\Omega |F(t_{j-1})(\omega)|^2(t_j-t_{j-1})\;\mathbb{P}(d\omega).\ \blacksquare$$

We can prove now the following result.

Theorem 2.2.7 *For any $F\in C_{ad}([0,T])$ there exists the limit*

$$\lim_{|\sigma|\to 0} I_\sigma(F)=:\int_0^T F(s)dB(s),\qquad(2.2.7)$$

in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Moreover we have

$$\int_\Omega\left(\int_0^T F(s)dB(s)\right)\mathbb{P}(d\omega)=0,\qquad(2.2.8)$$

and

$$\int_\Omega\left|\int_0^T F(s)dB(s)\right|^2\mathbb{P}(d\omega)=\int_\Omega\left(\int_0^T |F(s)|^2\,ds\right)\;\mathbb{P}(d\omega).\qquad(2.2.9)$$

The function of $L^2(\Omega,\mathcal{F},\mathbb{P})$, $\boxed{\int_0^T F(s)dB(s),}$ is called the *Itô integral* of the function F in $[0,T]$.

Proof of Theorem 2.2.7— For any $\sigma=\{0=t_0<t_1<\dots<t_n=T\}\in\Sigma$ we consider the function

$$\cancel{I_\sigma} F_\sigma=\sum_{j=1}^n F(t_{j-1})(B(t_j)-B(t_{j-1})).$$

$\int_0^T F(s)\,ds=\lim_{|\sigma|\to 0} J_\sigma(F)$, dove $J_\sigma(F)=\sum_{j=1}^{n} F(t_{k-1})(t_k-t_{k-1})$

In view of (2.2.9) we have, for any $\sigma, \eta \in \Sigma$

$$\int_\Omega |I_\sigma(F) - I_\eta(F)|^2 \mathbb{P}(d\omega) = \int_\Omega \left(\int_0^T (F_\sigma(t) - F_\eta(t))^2 dt \right) \mathbb{P}(d\omega).$$

Now the conclusion follows since

$$\lim_{|\sigma| \to 0} F_\sigma = F \text{ in } C_{ad}([0,T]),$$

as easily checked. ■

We can define in an obvious way the Itô integral $\int_a^b F(s)dB(s)$ in any interval $[a,b] \subset [0,T]$. We have

$$\int_\Omega \left(\int_a^b F(s)dB(s) \right) \mathbb{P}(d\omega) = 0, \tag{2.2.10}$$

and

$$\int_\Omega \left| \int_a^b F(s)dB(s) \right|^2 \mathbb{P}(d\omega) = \int_\Omega \left(\int_a^b |F(s)|^2 \, ds \right) \mathbb{P}(d\omega). \tag{2.2.11}$$

Corollary 2.2.8 *For any $F, G \in C_{ad}([0,T])$ and for any interval $[a,b] \subset [0,T]$ we have*

$$\begin{aligned} &\int_\Omega \left(\int_a^b F(s)dB(s) \int_a^b G(s)dB(s) \right) \mathbb{P}(d\omega) \\ &= \int_\Omega \left(\int_a^b F(s)G(s)ds \right) \mathbb{P}(d\omega). \end{aligned} \tag{2.2.12}$$

Proof — We have

$$\int_\Omega \left(\int_a^b F(s)dB(s) \int_a^b G(s)dB(s) \right) \mathbb{P}(d\omega)$$

$$= \frac{1}{2} \int_\Omega \left| \int_a^b (F(s) + G(s))dB(s) \right|^2 \mathbb{P}(d\omega)$$

$$-\frac{1}{2} \int_\Omega \left| \int_a^b F(s)dB(s) \right|^2 \mathbb{P}(d\omega) - \frac{1}{2} \int_\Omega \left| \int_a^b G(s)dB(s) \right|^2 \mathbb{P}(d\omega)$$

$$= \frac{1}{2} \int_\Omega \int_a^b |F(s) + G(s)|^2 \, ds \, \mathbb{P}(d\omega)$$

$$-\frac{1}{2} \int_\Omega \int_a^b |F(s)|^2 ds \, \mathbb{P}(d\omega) - \frac{1}{2} \int_\Omega \int_a^b |G(s)|^2 ds \, \mathbb{P}(d\omega)$$

$$= \int_\Omega \left(\int_a^b F(s)G(s)ds \right) \mathbb{P}(d\omega). \blacksquare$$

Example 2.2.9 Let us prove that

$$\int_0^T B(t)dB(t) = \frac{1}{2}(B^2(T) - T). \tag{2.2.13}$$

If $\sigma = \{0 = t_0 < t_1 < \dots < t_n = T\} \in \Sigma$ we have recalling Proposition 1.2.11

$$\begin{aligned} I_\sigma(B)(\omega) &= \sum_{k=1}^n \omega(t_{k-1})(\omega(t_k) - \omega(t_{k-1})) \\ &= \frac{1}{2} \sum_{k=1}^n (\omega^2(t_k) - \omega^2(t_{k-1})) - \frac{1}{2} \sum_{k=1}^n |\omega(t_k) - \omega(t_{k-1})|^2 \\ &= \frac{1}{2}\, \omega^2(T) - \frac{1}{2} \sum_{k=1}^n |\omega(t_k) - \omega(t_{k-1})|^2 \to \frac{1}{2}(\omega^2(T) - T), \end{aligned}$$

for $|\sigma| \to 0$, in $L^2(\Omega)$. $\blacksquare$

Arguing as in the previous example we find that

$$\lim_{|\sigma|\to 0} \sum_{k=1}^{n} \omega(t_k)(\omega(t_k) - \omega(t_{k-1})) = \frac{1}{2}\,(\omega^2(T) + T), \text{ in } L^2(\Omega).$$

$$\lim_{|\sigma|\to 0} \sum_{k=1}^{n} \frac{\omega(t_k) + \omega(t_{k-1})}{2}(\omega(t_k) - \omega(t_{k-1})) = \frac{1}{2}\,\omega^2(T), \text{ in } L^2(\Omega).$$

This shows that the definition of Itô integral depends on the choice of integral sums.

We want finally to study $\int_0^t F(s)dB(s)$ as a function of t. (l'integrale di Itô ha una dipendenza con dall'estremo superiore

Theorem 2.2.10 *Let $F \in C_{ad}([0,T])$ and let*

$$G(t) = \int_0^t F(s)dB(s), \; t \in [0,T].$$

Then $\boxed{G \in C_{ad}([0,T])}$ *and it holds*

$$\|G\|_{C_{ad}([0,T])} \le \sqrt{T}\,\|F\|_{C_{ad}([0,T])}. \tag{2.2.14}$$

Proof — It is easy to see that $G(t)$ is $\mathcal{F}_t$– measurable for all $t \in [0,T]$. Moreover if $0 \le s < t \le T$ by (2.2.11) we have

G(t) è limite di funzioni $\mathcal{F}_t$ ($\lim \sum F(t_{k-1})(B(t_k)$ C_{ad}

$$\int_\Omega |G(t) - G(s)|^2 \mathbb{P}(d\omega) = \int_\Omega \left| \int_s^t F(u)d\omega(u) \right|^2 \mathbb{P}(d\omega)$$

$$= \int_\Omega \left(\int_s^t |F(u)|^2 du \right) \mathbb{P}(d\omega) = \int_s^t \left(\int_\Omega |F(u)|^2 \mathbb{P}(d\omega) \right) du.$$

This implies that $G \in C([0,T]; L^2(\Omega))$ as claimed. We prove now (2.2.14). Let $t \in [0,T]$; again by (2.2.11) we have

perché $\|G(t) - G(s)\|^2_{C_{ad}}$ $\|F\|^2_{C_{ad}} \cdot |t$

$$\int_\Omega |G(t)|^2 \mathbb{P}(d\omega) = \int_0^t \int_\Omega |F(u)|^2 \mathbb{P}(d\omega)du \le T\|F\|^2_{C_{ad}([0,T])},$$

that yields (2.2.14) . ■

Remark 2.2.11 Under the assumptions of Theorem 2.2.10 one can show that function $t \mapsto G(t)(\omega)$, is continuous for almost all $\omega \in \Omega$, see Chapter 7.