Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. We consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 + \int_0^t e^{-\frac{s^2}{2}} X(s)\, ds + \int_0^t \sin s dB(s) \ , \tag{1}$$
$$dX(t) = e^{-\frac{t^2}{2}} X(t)\, dt + \sin t dB(t) \ ,$$

where $\{B(t)\}_{t\geq 0}$ *is the Brownian motion.*

1. *Solve the equation (1) assuming the initial datum* $X_0 = 0$
2. *Compute the distribution function of the r.v.* $X(t, X_0)$ *for any given* $t > 0$.
3. *Compute the characteristic function of the r.v.* $X(1,0)$, *where* $(X(t, X_0)\ , t\geq 0)$ *is the solution of the previous problem.*

Solution:

1. The equation (1) is an Itô SDE with additive noise. Setting

$$Y(t) = f(t, X(t)) := X(t) \exp\left\{-\int_0^t dse^{-\frac{s^2}{2}}\right\} \tag{2}$$

and computing the Itô differential of $Y(t)$ we get

$$df(t, X(t)) = \left\{(\partial_t f)(t, X(t)) + (\partial_x f)(t, X(s))\, e^{-\frac{t^2}{2}} X(t) + \frac{1}{2}\sin^2 t \left(\partial_{xx}^2 f\right)(t, X(t))\right\} dt \tag{3}$$
$$+ \sin t\, (\partial_x f)(t, X(t))\, dB(t) \ .$$

But, since $f(t,x) = x\exp\left\{-\int_0^t dse^{-\frac{s^2}{2}}\right\}$ and

$$\partial_t f(t,x) = -e^{-\frac{t^2}{2}} f(t,x) \ , \tag{4}$$
$$\partial_x f(t,x) = \frac{f(t,x)}{x} \ , \tag{5}$$
$$\partial_x^2 f(t,x) = 0 \ , \tag{6}$$

we have

$$dY(t) = \exp\left\{-\int_0^t ds e^{-\frac{s^2}{2}}\right\} \sin t dB(t) \tag{7}$$

that is, taking into account that $Y(0, X(0)) = X_0$,

$$Y(t, X_0) = X_0 + \int_0^t \exp\left\{-\int_0^t ds e^{-\frac{s^2}{2}}\right\} \sin s dB(s) \ , \tag{8}$$

which implies

$$\begin{aligned} X(t, X_0) &= \exp\left\{\int_0^t ds e^{-\frac{s^2}{2}}\right\} Y(t, X_0) \qquad (9) \\ &= \exp\left\{\int_0^t ds e^{-\frac{s^2}{2}}\right\} \left[X_0 + \int_0^t \exp\left\{-\int_0^s d\tau e^{-\frac{\tau^2}{2}}\right\} \sin s dB(s)\right] \ . \end{aligned}$$

2. Taking $X_0 = 0$, we obtain

$$X(t, 0) = \int_0^t \exp\left\{\int_s^t d\tau e^{-\frac{\tau^2}{2}}\right\} \sin s dB(s) \tag{10}$$

wich is a Gaussian r.v. with expectation

$$\mathbb{E}[X(t, 0)] = \mathbb{E}\left[\int_0^t \exp\left\{\int_s^t d\tau e^{-\frac{\tau^2}{2}}\right\} \sin s dB(s)\right] = 0 \tag{11}$$

and variance

$$\mathbb{E}\left[X^2(s, 0)\right] = \int_0^t ds \exp 2\left\{\int_s^t d\tau e^{-\frac{\tau^2}{2}}\right\} \sin^2 s \ . \tag{12}$$

3. The r.v.

$$X(1, 0) = \int_0^1 \exp\left\{\int_s^1 d\tau e^{-\frac{\tau^2}{2}}\right\} \sin s dB(s) \tag{13}$$

has Gaussian distribution with expectation

$$\mu = 0 \tag{14}$$

and variance

$$\sigma^2 := \int_0^1 ds \exp 2\left\{\int_s^1 d\tau e^{-\frac{\tau^2}{2}}\right\} \sin^2 s \ . \tag{15}$$

Hence the characteristic function of $X(1, 0)$ is

$$\varphi_{X(1,0)} = e^{-\frac{1}{2}t^2\sigma^2} \ . \tag{16}$$

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Exercise 2 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\{X_k\}_{k\geq 0}$ *a sequence of r.v.'s such that if* $\{\mathcal{F}_n\}_{n\geq 0}$ *is the natural filtration generated by* $\{X_k\}_{k\geq 0}, \forall k \geq 0, B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}[X_{k+1} \in B|\mathcal{F}_k] = \mathbb{P}[X_{k+1} \in B|X_k] = \int_B dx \frac{e^{-\frac{(x-X_k)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \ .$$

1. (a) Prove that the function $\mathbb{R} \ni x \longmapsto f(x) := e^{\frac{\alpha}{\sigma^2}x} \in \mathbb{R}$ is such that $\forall k \geq 0, \mathbb{E}[f(X_{k+1})|\mathcal{F}_k] = \lambda f(X_k)$ and compute λ.
   (b) If $X_0 \stackrel{d}{=} N(0,1)$, determine a numerical sequence $\{c_n\}_{n\geq 0}$ such that the sequence of r.v.'s $\{Y_n\}_{n\geq 0}$ so defined $\forall n \geq 0, Y_n := c_n f(X_n)$ is a martingale w.r.t. the filtration $\{\mathcal{F}_n\}_{n\geq 0}$.
   (c) Study the convergence of $\{Y_n\}_{n\geq 1}$.

Solution:

1. Since
$$\mathbb{E}[f(X_{k+1})|\mathcal{F}_k] = \int_{\mathbb{R}} dx \frac{e^{-\frac{(x-X_k)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} e^{\frac{\alpha}{\sigma^2}x} = e^{\frac{\alpha^2}{2\sigma^2}} e^{\frac{\alpha}{\sigma^2}X_k}$$
we get $\lambda = e^{\frac{\alpha^2}{2\sigma^2}}$.

2. Let us set $\forall n \geq 0, c_n := \lambda^{-n}$. Then
$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \lambda^{-(n+1)}\mathbb{E}[f(X_{n+1})|X_n] = \lambda^{-n}X_n = Y_n \ .$$
Hence $\mathbb{E}[Y_n] = \mathbb{E}[Y_0] = \mathbb{E}[X_0] = e^{\frac{\alpha^2}{2\sigma^4}}$. Therefore, $\{Y_n\}_{n\geq 1}$ is a martingale.

3. Since $\{Y_n\}_{n\geq 1}$ is a non-negative martingale it converges $\mathbb{P}-a.s.$.

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