## Solution of the execises of the exam of the course Probability and Stochastic Processes

## a.y. 2023/202402/02/2024

**Exercise 1** Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$  be the filtered Wiener space. We consider the stochastic process described by the Itô Stochastic Differential Equation

$$X(t, X_0) = X_0 + \int_0^t ds X(s) + \int_0^t \sqrt{2} s X(s) \, dB(s) \quad , \tag{1}$$
$$dX(t) = X(t) \, dt + \sqrt{2} t X(t) \, dB(t) \quad .$$

where  $\{B(t)\}_{t\geq 0}$  is the Brownian motion.

- 1. Solve the equation (1) assuming the initial datum  $X_0 = 1$ .
- 2. Compute the probability density of the r.v.  $X(t, X_0), t \ge 0$ .
- 3. Compute the density of the random vector  $(\log X(2, X_0), \log X(1, X_0))$ .

## Solution:

1. The equation (1) is an Itô EDS with moltiplicative noise. Setting

$$Y(t) = f(t, X(t)) := \log \frac{X(t)}{X_0}$$

and computing the Itô's differential of Y(t), since  $f(t, x) = \log x$  and

$$\begin{split} \partial_t f\left(t,x\right) &= 0 \ , \\ \partial_x f\left(t,x\right) &= \frac{1}{x} \ , \\ \partial_x^2 f\left(t,x\right) &= -\frac{1}{x^2} \end{split}$$

,

we get

$$dY(t) = (1 - t^{2}) dt + \sqrt{2}t dB(t) ,$$
  
$$Y(t) = \int_{0}^{t} ds (1 - s^{2}) + \int_{0}^{t} \sqrt{2}s dB(s) .$$

That is, taking into account that Y(0, X(0)) = 0,

$$X(t, X_0) = X_0 e^{t - \frac{t^3}{3} + \sqrt{2} \int_0^t s dB(s)} .$$
<sup>(2)</sup>

2. Since  $X_0 = 1$ , from the previous equation we have that  $\mathbb{P}\left\{X\left(t\right) \leq x\right\}$ , where  $X\left(t\right) := X\left(t,1\right)$  is equal to  $\mathbb{P}\left\{\int_0^t s dB\left(s\right) \leq \frac{\log x + \frac{t^3}{3} - t}{\sqrt{2}}\right\}$ , but  $\int_0^t s dB\left(s\right)$  is a Gaussian centered r.v. with variance  $\frac{t^3}{3}$ , hence

$$f_{X(t)} = \frac{1}{2} \sqrt{\frac{3}{\pi t^3}} \frac{1}{x} \exp\left\{-\frac{3}{4t^3} \left(\log x + \frac{t^3}{3} - t\right)^2\right\}$$

3. From (2) it follows that

$$\mathbb{E} [Y (t, X_0)] = t - \frac{t^3}{3}$$
  

$$Cov [Y (t, X_0), Y (s, X_0)] = \mathbb{E} [(Y (t, X_0) - \mathbb{E} [Y (t, X_0)]) (Y (s, X_0) - \mathbb{E} [Y (s, X_0)])]$$
  

$$= 2\mathbb{E} \left[ \int_0^{t \wedge s} d\tau \tau^2 \right] = \frac{2}{3} (t \wedge s)^3$$

The random vector

$$Y := (\log X(2, X_0), \log X(1, X_0))$$
$$= \left(2 - \frac{8}{3} + \sqrt{2} \int_0^2 t dB(t), 1 - \frac{1}{3} + \sqrt{2} \int_0^1 t dB(t)\right)$$

has Gaussian distribution with expectation vector

$$\mu = \left(-\frac{2}{3}, \frac{2}{3}\right) = (\mu_1, \mu_2)$$

and covariance matrix

$$C := \begin{pmatrix} \frac{16}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} a & b \\ b & b \end{pmatrix} .$$

Therefore,

$$f_{Y}(x,y) = \frac{1}{\sqrt{(2\pi)^{2} \det C}} \exp\left\{-\frac{\langle C^{-1}(x-\mu_{1},y-\mu_{2}),(x-\mu_{1},y-\mu_{2})\rangle}{2}\right\}$$
(3)  
$$= \frac{3}{8\pi\sqrt{2}} \exp\left\{-\frac{1}{2}\left[\frac{3}{14}\left(x+\frac{2}{3}\right)^{2}-\frac{6}{14}\left(x+\frac{2}{3}\right)\left(y-\frac{2}{3}\right)+\frac{12}{7}\left(y-\frac{2}{3}\right)^{2}\right]\right\}.$$

**Exercise 2** Let  $\{X_i\}_{i\geq 1}$  a sequence of r.v.'s on a sample space  $(\Omega, \mathcal{F}, \mathbb{P}), \{f_i\}_{i\geq 1}$  be a sequence of given functions such that  $f_i : \mathbb{R}^i \mapsto \mathbb{R}$ .

- 1. What are the minimal requirement on the terms of the sequences  $\{X_i\}_{i\geq 1}$  and  $\{f_i\}_{i\geq 1}$  have to satisfy for the sequence  $\{Y_i\}_{i\geq 0}$  such that  $\forall i \geq 1, Y_i := \left(\sum_{j=1}^i X_j f_{j-1}(X_1, ..., X_{j-1})\right)^2$  to be a submartingale w.r.t. the natural filtration  $\{\mathcal{F}_i\}_{i\geq 0}$  generated by  $\{X_i\}_{i\geq 1}$ ?
- 2. Supposing that the requirements asked above are satisfied, write the Doob-Meyer decomposition of  $\{Y_i\}_{i\geq 1}$ .

**Solution:** Let, for any  $i \ge 1, S_i := \sum_{j=1}^i X_j f_{j-1}(X_1, ..., X_{j-1}) \cdot \mathbb{E}\left[|S_i|\right] \le \sup_{1 \le j \le i-1} \sup_{(x_1, ..., x_{j-1}) \in \mathbb{R}^{j-1}} |f_{j-1}(x_1, ..., x_{j-1})| = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) \cdot \mathbb{E}\left[|S_i|\right] \le \sup_{1 \le j \le i-1} \sup_{(x_1, ..., x_{j-1}) \in \mathbb{R}^{j-1}} |f_{j-1}(x_1, ..., x_{j-1})| = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) \cdot \mathbb{E}\left[|S_i|\right] \le \sup_{1 \le j \le i-1} \sup_{(x_1, ..., x_{j-1}) \in \mathbb{R}^{j-1}} |f_{j-1}(x_1, ..., x_{j-1})| = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) \cdot \mathbb{E}\left[|S_i|\right] \le \sup_{1 \le j \le i-1} \sup_{(x_1, ..., x_{j-1}) \in \mathbb{R}^{j-1}} |f_{j-1}(x_1, ..., x_{j-1})| = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) \cdot \mathbb{E}\left[|S_i|\right] \le \sup_{1 \le j \le i-1} \sup_{(x_1, ..., x_{j-1}) \in \mathbb{R}^{j-1}} |f_{j-1}(x_1, ..., x_{j-1})| = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) \cdot \mathbb{E}\left[|S_i|\right] \le \sup_{1 \le i \le j \le i-1} \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..., X_{j-1}) = \sum_{i=1}^{j} X_i f_{j-1}(X_1, ..$ 

$$\mathbb{E}\left[S_{i+1}|\mathcal{F}_i\right] = \mathbb{E}\left[S_i + X_{i+1}W_i|\mathcal{F}_i\right] = S_i + W_i\mathbb{E}\left[X_{i+1}|\mathcal{F}_i\right]$$

Hence, if, for any  $i \geq 1, \mathbb{E}[|X_i|] < \infty, W_{i-1}\mathbb{E}[X_i|\mathcal{F}_{i-1}] \geq 0$  and  $f_i$  is bounded, then  $\{S_i\}_{i\geq 1}$  is a submartingale w.r.t.  $\{\mathcal{F}_i\}_{i\geq 1}$ .

Since  $x \mapsto x^2$  is a convex function, by the Jensen inequality

$$\mathbb{E}\left[Y_{i+1}|\mathcal{F}_i\right] = \mathbb{E}\left[\left(S_{i+1}\right)^2|\mathcal{F}_i\right] \ge \left(\mathbb{E}\left[S_{i+1}|\mathcal{F}_i\right]\right)^2 \ge \left(S_i\right)^2 = Y_i ,$$

that is  $\{Y_i\}_{i\geq 1}$  is a submartingale. The Doob-Meyer decomposition of  $\{Y_i\}_{i\geq 0}$  is then given by the martingale  $\{\overline{m}_i\}_{i\geq 1}$  where  $\forall i\geq 1, m_i:=Y_i-A_i$  and by the compensator  $\{\overline{A}_i\}_{i\geq 1}$  where  $\forall i\geq 1$ ,

$$A_{i} := X_{1}f_{0} + \sum_{j=1}^{i} \left(\mathbb{E}\left[Y_{j}|\mathcal{F}_{j-1}\right] - Y_{j-1}\right)$$
  
$$= X_{1}f_{0} + \sum_{j=1}^{i} \left(\mathbb{E}\left[Y_{j} - Y_{j-1}|\mathcal{F}_{j-1}\right]\right) =$$
  
$$= X_{1}f_{0} + \sum_{j=1}^{i} \left(\mathbb{E}\left[\left(S_{j-1} + X_{j}W_{j-1}\right)^{2} - S_{j-1}^{2}|\mathcal{F}_{j-1}\right]\right)$$
  
$$= X_{1}f_{0} + \sum_{j=1}^{i} \left(\mathbb{E}\left[X_{j}W_{j-1}\left(2S_{j-1} + X_{j}W_{j-1}\right)|\mathcal{F}_{j-1}\right]\right) .$$

Assuming that  $\{S_i\}_{i>1}$  is a martingale i.e.  $\forall i \ge 1, \mathbb{E}[X_i | \mathcal{F}_{i-1}] = 0$ ,

$$A_i = X_1 f_0 + \sum_{j=2}^{i} \mathbb{E} \left[ X_j^2 | \mathcal{F}_{j-1} \right] W_{j-1} .$$