Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. We consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 + \int_0^t ds X(s) + \int_0^t \sqrt{2} s X(s) dB(s) \ , \tag{1}$$

$$dX(t) = X(t) dt + \sqrt{2} t X(t) dB(t) \ .$$

where $\{B(t)\}_{t\geq 0}$ *is the Brownian motion.*

1. *Solve the equation (1) assuming the initial datum* $X_0 = 1$.
2. *Compute the probability density of the r.v.* $X(t, X_0), t \geq 0$.
3. *Compute the density of the random vector* $(\log X(2, X_0), \log X(1, X_0))$.

Solution:

1. The equation (1) is an Itô EDS with moltiplicative noise.
   Setting
   $$Y(t) = f(t, X(t)) := \log \frac{X(t)}{X_0}$$
   and computing the Itô's differential of $Y(t)$, since $f(t,x) = \log x$ and
   $$\partial_t f(t,x) = 0 \ ,$$
   $$\partial_x f(t,x) = \frac{1}{x} \ ,$$
   $$\partial_x^2 f(t,x) = -\frac{1}{x^2} \ ,$$
   we get
   $$dY(t) = \left(1 - t^2\right) dt + \sqrt{2} t dB(t) \ ,$$
   $$Y(t) = \int_0^t ds \left(1 - s^2\right) + \int_0^t \sqrt{2} s dB(s) \ .$$

That is, taking into account that $Y(0, X(0)) = 0$,

$$X(t, X_0) = X_0 e^{t - \frac{t^3}{3} + \sqrt{2}\int_0^t s dB(s)} .$$

2. Since $X_0 = 1$, from the previous equation we have that $\mathbb{P}\{X(t) \leq x\}$, where $X(t) := X(t, 1)$ is equal to $\mathbb{P}\left\{\int_0^t s dB(s) \leq \frac{\log x + \frac{t^3}{3} - t}{\sqrt{2}}\right\}$, but $\int_0^t s dB(s)$ is a Gaussian centered r.v. with variance $\frac{t^3}{3}$, hence

$$f_{X(t)} = \frac{1}{2}\sqrt{\frac{3}{\pi t^3}}\frac{1}{x}\exp\left\{-\frac{3}{4t^3}\left(\log x + \frac{t^3}{3} - t\right)^2\right\}$$

3. From (**??**) it follows that

$$\begin{aligned}
\mathbb{E}[Y(t, X_0)] &= t - \frac{t^3}{3} \\
Cov[Y(t, X_0), Y(s, X_0)] &= \mathbb{E}[(Y(t, X_0) - \mathbb{E}[Y(t, X_0)])(Y(s, X_0) - \mathbb{E}[Y(s, X_0)])] \\
&= 2\mathbb{E}\left[\int_0^{t\wedge s} d\tau \tau^2\right] = \frac{2}{3}(t \wedge s)^3
\end{aligned}$$

The random vector

$$\begin{aligned}
Y \quad &: = (\log X(2, X_0), \log X(1, X_0)) \\
&= \left(2 - \frac{8}{3} + \sqrt{2}\int_0^2 t dB(t), 1 - \frac{1}{3} + \sqrt{2}\int_0^1 t dB(t)\right)
\end{aligned}$$

has Gaussian distribution with expectation vector

$$\mu = \left(-\frac{2}{3}, \frac{2}{3}\right) = (\mu_1, \mu_2)$$

and covariance matrix

$$C := \begin{pmatrix} \frac{16}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} a & b \\ b & b \end{pmatrix} .$$

Therefore,

$$\begin{aligned}
f_Y(x, y) &= \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left\{-\frac{\langle C^{-1}(x - \mu_1, y - \mu_2), (x - \mu_1, y - \mu_2)\rangle}{2}\right\} \qquad (2) \\
&= \frac{3}{8\pi\sqrt{2}} \exp\left\{-\frac{1}{2}\left[\frac{3}{14}\left(x + \frac{2}{3}\right)^2 - \frac{6}{14}\left(x + \frac{2}{3}\right)\left(y - \frac{2}{3}\right) + \frac{12}{7}\left(y - \frac{2}{3}\right)^2\right]\right\} .
\end{aligned}$$

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Exercise 2 *Let* $\{X_i\}_{i\geq 1}$ *a sequence of r.v.'s on a sample space* $(\Omega, \mathcal{F}, \mathbb{P})$, $\{f_i\}_{i\geq 1}$ *be a sequence of given functions such that* $f_i : \mathbb{R}^i \mapsto \mathbb{R}$.

1. *What are the minimal requirement on the terms of the sequences* $\{X_i\}_{i\geq 1}$ *and* $\{f_i\}_{i\geq 1}$ *have to satisfy for the sequence* $\{Y_i\}_{i\geq 0}$ *such that* $\forall i \geq 1, Y_i := \left(\sum_{j=1}^{i} X_j f_{j-1}(X_1, .., X_{j-1})\right)^2$ *to be a submartingale w.r.t. the natural filtration* $\{\mathcal{F}_i\}_{i\geq 0}$ *generated by* $\{X_i\}_{i\geq 1}$*?*
2. *Supposing that the requirements asked above are satisfied, write the Doob-Meyer decomposition of* $\{Y_i\}_{i\geq 1}$ *.*

Solution: Let, for any $i \geq 1, S_i := \sum_{j=1}^{i} X_j f_{j-1}(X_1, .., X_{j-1}) .\mathbb{E}[|S_i|] \leq \sup_{1\leq j\leq i-1} \sup_{(x_1,..,x_{j-1})\in\mathbb{R}^{j-1}} |f_{j-1}(x_1, .., x_{j-1})$
Moreover, since $S_{i+1} = S_i + X_{i+1}W_i$ where $\forall i \geq 1, W_i := f_i(X_1, .., X_i)$,

$$\mathbb{E}[S_{i+1}|\mathcal{F}_i] = \mathbb{E}[S_i + X_{i+1}W_i|\mathcal{F}_i] = S_i + W_i\mathbb{E}[X_{i+1}|\mathcal{F}_i] \ .$$

Hence, if, for any $i \geq 1, \mathbb{E}[|X_i|] < \infty, W_{i-1}\mathbb{E}[X_i|\mathcal{F}_{i-1}] \geq 0$ and f_i is bounded, then $\{S_i\}_{i\geq 1}$ is a submartingale w.r.t. $\{\mathcal{F}_i\}_{i\geq 1}$.

Since $x \mapsto x^2$ is a convex function, by the Jensen inequality

$$\mathbb{E}[Y_{i+1}|\mathcal{F}_i] = \mathbb{E}\left[(S_{i+1})^2|\mathcal{F}_i\right] \geq (\mathbb{E}[S_{i+1}|\mathcal{F}_i])^2 \geq (S_i)^2 = Y_i \ ,$$

that is $\{Y_i\}_{i\geq 1}$ is a submartingale. The Doob-Meyer decomposition of $\{Y_i\}_{i\geq 0}$ is then given by the martingale $\{m_i\}_{i\geq 1}$ where $\forall i \geq 1, m_i := Y_i - A_i$ and by the compensator $\{A_i\}_{i\geq 1}$ where $\forall i \geq 1$,

$$\begin{aligned} A_i \ : &= X_1 f_0 + \sum_{j=1}^{i} (\mathbb{E}[Y_j|\mathcal{F}_{j-1}] - Y_{j-1}) \\ &= X_1 f_0 + \sum_{j=1}^{i} (\mathbb{E}[Y_j - Y_{j-1}|\mathcal{F}_{j-1}]) = \\ &= X_1 f_0 + \sum_{j=1}^{i} \left(\mathbb{E}\left[(S_{j-1} + X_j W_{j-1})^2 - S_{j-1}^2|\mathcal{F}_{j-1}\right]\right) \\ &= X_1 f_0 + \sum_{j=1}^{i} (\mathbb{E}[X_j W_{j-1}(2S_{j-1} + X_j W_{j-1})|\mathcal{F}_{j-1}]) \ . \end{aligned}$$

Assuming that $\{S_i\}_{i\geq 1}$ is a martingale i.e. $\forall i \geq 1, \mathbb{E}[X_i|\mathcal{F}_{i-1}] = 0$,

$$A_i = X_1 f_0 + \sum_{j=2}^{i} \mathbb{E}\left[X_j^2|\mathcal{F}_{j-1}\right] W_{j-1} \ .$$

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