Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let $\{X_i\}_{i\geq 0}$ a sequence of i.i.d.r.v.'s on a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\geq 0}, \mathbb{P}\right)$ such that $(\Omega, \mathcal{F}) = (\mathbb{Z}_+, \mathcal{P}(\mathbb{Z}_+))$, $\mathcal{F}_0$ is the trivial σ algebra and $\forall n \geq 1, \mathcal{F}_n := \sigma(\{1\}, .., \{n\}, \{k \in \Omega : k \geq n+1\})$, $\mathbb{P}\{n\} = \frac{1}{n} - \frac{1}{n+1}$ and $\forall i \geq 0, X_i := (i+1)\mathbf{1}_{\{k\in\Omega:k\geq n+1\}}$. Prove that $\{X_n\}_{n\geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$ and study its convergence.*

Solution: By the definition of $\mathbb{P}, \forall n \geq 0, \mathbb{P}\{k \in \Omega : k \geq n+1\} = \frac{1}{n+1}$. Hence,

$$\begin{aligned} \mathbb{E}[X_{n+1}|\mathcal{F}_n] &= (n+2)\,\mathbb{P}(\{k \in \Omega : k \geq n+2\} \mid \{k \in \Omega : k \geq n+1\})\,\mathbf{1}_{\{k\in\Omega:k\geq n+1\}} \\ &= (n+2)\,\frac{\frac{1}{n+2}}{\frac{1}{n+1}}\mathbf{1}_{\{k\in\Omega:k\geq n+1\}} = X_n \ . \end{aligned}$$

Moreover, $\forall n \geq 0, X_n > 0$ so that $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] = \mathbb{E}[X_0] = 1$. Therefore $\{X_i\}_{i\geq 0}$ is a L^1-martingale hence convergent $\mathbb{P}-q.c.$ to a L^1 r.v. ξ. Moreover, $\forall n \geq 0$,

$$\varphi_{X_n}(t) = \mathbb{E}\left[e^{itX_n}\right] = \mathbb{P}\{k \in \Omega : k \leq n\} + \frac{e^{it(n+1)}}{n+1} \ .$$

Therefore, $\varphi_{X_n}(0) = 1$ and its pointwise limit as $n \to \infty$ is 1, that is $\xi = 0\ \mathbb{P}-q.c.$. ∎

Exercise 2 *Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$\begin{aligned} X(t, X_0) &= X_0 + \int_0^t ds\,(\cos s)\,(X(s)+1) + \int_0^t \sin s\, dB(s) \ , \qquad (1) \\ dX(t) &= (\cos t)\,(X(t)+1)\,dt + (\sin t)\,dB(t) \ . \end{aligned}$$

where $(B(t), t \geq 0)$ is the Brownian motion.

1. *Compute the covariance of the stochastic process $(X(t,0)\ ,\ t \geq 0)$.*

Solution: The equation (1) sia a linear Itô's SDE with additive noise. Hence, considering the process

$$Y(t) = f(t, X(t)) = e^{-\int_0^t ds \cos s} X(t) = e^{-\sin t} X(t),$$

by the Itô's lemma we get

$$dY(t) = (\cos t) e^{-\sin t} dt + (\sin t) e^{-\sin t} dB(s)$$

whose solution is

$$Y(t) = X_0 + \int_0^t ds e^{-\sin s} \cos s + \int_0^t e^{-\sin s} \sin s dB(s) \ .$$

Therefore,

$$X(t) = X_0 e^{\sin t} + \int_0^t ds e^{\sin t - \sin s} \cos s + \int_0^t e^{\sin t - \sin s} \sin s dB(s) \ .$$

1.

$$\mathbb{E}\left[(X(t,0) - \mathbb{E}[X(t,0)])(X(s,0) - \mathbb{E}[X(s,0)])\right] =$$
$$= e^{(\sin t) + (\sin s)} \int_0^{t \wedge s} d\tau e^{-2\sin \tau} (\sin \tau)^2$$

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Exercise 3 *Compute the probability density function of the random variable* $\exp X(t,0)$, *where* $(X(t,0)\ ,\ t \geq 0)$ *is the solution of the Itô SDE (1).*

Solution: For any $t > 0, X(t,0) \overset{d}{=} N(\mu(t), \sigma^2(t))$, where $\mu(t) = \int_0^t ds e^{\sin t - \sin s} \cos s$ and $\sigma^2(t) = \int_0^t ds e^{2(\sin t - \sin s)} (\sin s)^2$. Thus, denoting by $Z := \exp X(t,0), \forall z > 0$,

$$\mathbb{P}\{Z \leq z\} = \mathbb{P}\{X(t,0) \leq \log z\} = \int_{-\infty}^{\log z} dx \frac{e^{-\frac{(x-\mu(t))^2}{2\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}} \ ,$$

which implies

$$f_Z(z) = \frac{d}{dz}\mathbb{P}\{Z \leq z\} = \frac{e^{-\frac{(\log z - \mu(t))^2}{2\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}} \frac{1}{z} \mathbf{1}_{(0,+\infty)}(z) \ .$$

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