Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 + \int_0^t ds \frac{s}{2} X(s) + \int_0^t \sqrt{s} X(s) \, dB(s) \ , \tag{1}$$
$$dX(t) = \frac{t}{2} X(t) \, dt + \sqrt{t} X(t) \, dB(t) \ .$$

where $(B(t), t \geq 0)$ *is the Brownian motion.*

1. *Compute the variance of the stochastic process* $(X(t,1)\ ,\ t \geq 0)$.

Solution: The equation (1) sia a Itô's SDE with multiplicative noise.

1. Setting

$$Y(t) = f(t, X(t)) := \log \frac{X(t)}{X_0} \tag{2}$$

and computing the Itô's differential of $Y(t)$, since $f(t,x) = \log \frac{x}{X_0}$ and

$$\partial_t f(t,x) = 0 \ , \tag{3}$$
$$\partial_x f(t,x) = \frac{X_0}{x} \ , \tag{4}$$
$$\partial_x^2 f(t,x) = -\frac{X_0}{x^2} \ , \tag{5}$$

we get

$$dY(t) = \sqrt{t} dB(t) \ ,\ Y(t) = \int_0^t \sqrt{s} dB(s) \ . \tag{6}$$

That is, taking into account that $Y(0, X(0)) = 0$,

$$X(t, X_0) = X_0 e^{\int_0^t \sqrt{s} dB(s)} \ . \tag{7}$$

2. Setting $X_0 = 1$, from (1) we get

$$\mathbb{E}\left[X\left(t,1\right)\right] = 1 + \mathbb{E}\left[\int_0^t ds \frac{s}{2} X\left(s\right)\right] + \mathbb{E}\left[\int_0^t \sqrt{s} X\left(s\right) dB\left(s\right)\right] \ , \tag{8}$$

that is, $\mu_X\left(t\right) := \mathbb{E}\left[X\left(t,1\right)\right]$, satisfies the Chauchy's problem

$$\begin{cases} \frac{d}{dt}\mu_X\left(t\right) = \frac{t}{2}\mu_X\left(t\right) \\ \mu_X\left(0\right) = 1 \end{cases} . \tag{9}$$

Hence,

$$\mu_X\left(t\right) = e^{\int_0^t ds \frac{s}{2}} = e^{\frac{t^2}{4}} \ . \tag{10}$$

Moreover, putting $Y\left(t\right) := f\left(t, X\left(t\right)\right) = X^2\left(t\right)$ and computing its Itô's differential we have

$$Y\left(t\right) = 1 + 2\int_0^t ds s X^2\left(s\right) + \int_0^t 2X^2\left(s\right)\sqrt{s} dB\left(s\right) \ . \tag{11}$$

Thus, denoting by $q_X\left(t\right) := \mathbb{E}\left[Y\left(t\right)\right] = \mathbb{E}\left[X^2\left(t\right)\right]$, taking the expectation of both sides of the preceding Itô's equations and taking the derivative w.r.t. t we obtain that q_X is the solution of the Cauchy's problem

$$\begin{cases} \frac{d}{dt}q_X\left(t\right) = 2tq_X\left(t\right) \\ q_X\left(0\right) = 1 \end{cases} \tag{12}$$

i.e.

$$q_X\left(t\right) = e^{t^2} \ . \tag{13}$$

Therefore, the variance of $X\left(t,1\right)$ is

$$\mathbb{E}\left[X^2\left(t,1\right)\right] - \mathbb{E}^2\left[X\left(t,1\right)\right] = q_X\left(t\right) - \mu_X^2\left(t\right) = e^{t^2} - e^{\frac{t^2}{2}} \ . \tag{14}$$

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Exercise 2 *Compute the characteristic function of the random vector* $\left(\log Y\left(1,1\right), \log Y\left(2,1\right)\right)$, *where* $\left(Y\left(t,1\right)\ ,\ t \geq 0\right)$ *is the stochastic process solution of the homogeneuous Itô SDE associated to the equation (1).*

Solution: The distribution of the random vector

$$\begin{aligned} Y &:= \left(\log X\left(1,1\right), \log X\left(2,1\right)\right) \\ &= \left(\int_0^1 \sqrt{t} dB\left(t\right), \int_0^2 \sqrt{t} dB\left(t\right)\right) \end{aligned} \tag{15}$$

is Gaussian with parameters

$$\mu = \left(0,0\right) \tag{16}$$

and, since

$$a\left(t\right) := \int_0^t ds s = \frac{t^2}{2} \ , \tag{17}$$

covariance matrix

$$C := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \ . \tag{18}$$

Thus, $\forall \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$,

$$\begin{aligned} \varphi_Y(\lambda) \quad &: \quad = \mathbb{E}\left[e^{i\langle \lambda, Y\rangle}\right] = e^{-\frac{1}{2}\langle C\lambda, \lambda\rangle} \qquad (19) \\ &= \quad \exp\left[-\frac{1}{2}\left(\frac{1}{2}\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2\right)\right] \ . \end{aligned}$$

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