# Solution of the execises of the exam of the course *Probability and Stochastic Processes*

a.y.2023/2024
06/19/2024

**Exercise 1** *Let* $\left\{\xi_i^{(n)}\right\}_{1\leq i\leq n, n\geq 1}$ *be a triangular array of independent. r.v.'s such that* $\mathbb{E}\left[\xi_k^{(n)}\right]=0$ *and* $\mathbb{E}\left[\left(\xi_k^{(n)}\right)^2\right]=\frac{1}{n^{2\alpha}}$ *with* $\alpha>0$. *Prove that the sequence of r.v.'s* $\{\zeta_n\}_{n\geq 0}$ *such that* $\zeta_0:=0,\zeta_1:=\xi_1^{(1)}$ *and* $\forall n\geq 1,\zeta_n:=\sum_{k=2}^n\varphi\left(\eta_{k-2},\eta_{k-1}\right)\eta_k$, *where* $\eta_k:=\sum_{i=1}^k\xi_i^{(k)}$ *and* $\varphi:\mathbb{R}^2\rightarrow\mathbb{R}$ *is a bounded function, is a martingale w.r.t. the filtration* $\{\mathcal{F}_n\}_{n\geq 0}$ *such that* $\mathcal{F}_0:=\{\varnothing,\Omega\}$ *and* $\forall n\geq 1,\mathcal{F}_n:=\sigma\left(\xi_1^{(1)},..,\xi_n^{(n)}\right)$. *Which are the values* $\alpha$ *that make it a convergent martingale?*

**Solution:** Let us denote by $M:=\sup_{(x,y)\in\mathbb{R}^2}|\varphi(x,y)|$.

$$\mathbb{E}\left[|\zeta_1|\right]=\mathbb{E}\left[\left|\xi_1^{(1)}\right|\right]\leq\mathbb{E}\left[\left(\xi_1^{(1)}\right)^2\right]=1$$

$\forall n\geq 2$ and $\alpha>0$,

$$\begin{aligned}\mathbb{E}\left[|\zeta_n|\right] &\leq M\sum_{k=2}^n\mathbb{E}\left[|\eta_k|\right]\leq M\sum_{k=2}^n\mathbb{E}\left[\left|\sum_{i=1}^k\xi_i^{(k)}\right|\right] \\ &\leq M\sum_{k=2}^n\sqrt{\sum_{i=1}^k\mathbb{E}\left[\left(\xi_i^{(k)}\right)^2\right]}\leq M\sum_{k=2}^n\sqrt{\sum_{i=1}^k\frac{1}{k^{2\alpha}}} \\ &\leq M\sum_{k=1}^n\frac{1}{k^{\frac{2\alpha-1}{2}}}<\infty\ ,\end{aligned}\tag{1}$$

Moreover, by definition $\mathcal{F}_n=\sigma\left(\eta_1,..,\eta_n\right)$ and $\zeta_{n+1}=\zeta_n+\varphi\left(\eta_{n-1},\eta_n\right)\eta_{n+1}$. Hence,

$$\mathbb{E}\left[\zeta_1|\mathcal{F}_0\right]=0=\zeta_0$$

and $\forall n\geq 1$,

$$\begin{aligned}\mathbb{E}\left[\zeta_{n+1}|\mathcal{F}_n\right] &= \zeta_n+\mathbb{E}\left[\varphi\left(\eta_{n-1},\eta_n\right)\eta_{n+1}|\mathcal{F}_n\right] \\ &= \zeta_n+\varphi\left(\eta_{n-1},\eta_n\right)\mathbb{E}\left[\eta_{n+1}|\mathcal{F}_n\right] \\ &= \zeta_n+\varphi\left(\eta_{n-1},\eta_n\right)\mathbb{E}\left[\eta_{n+1}\right]=\zeta_n\ .\end{aligned}$$

By (1) $\sup_{n\geq 0}\mathbb{E}\left[|\zeta_n|\right]$ is finite for $\alpha>\frac{3}{2}$, hence $\{\zeta_n\}_{n\geq 0}$ is a $L^1$-martingale. ■

**Exercise 2** *Let* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ *be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$X\left(t,X_0\right)=X_0+\int_0^t dss\left(X\left(s\right)+1\right)+\int_0^t\left(X\left(s\right)+s\right)dB\left(s\right)\ , \tag{2}$$
$$dX\left(t\right)=t\left(X\left(t\right)+1\right)dt+\left(X\left(t\right)+t\right)dB\left(t\right)\ .$$

*where* $\left(B\left(t\right),t\geq 0\right)$ *is the Brownian motion.*

**Solution:** (2) is a linear Itô SDE. Considering the associated homogeneous equation with initial datum equal to 1,

$$Y\left(t\right)=1+\int_0^t dssX\left(s\right)+\int_0^t X\left(s\right)dB\left(s\right)\ ,$$

whose solution is

$$\begin{aligned} Y\left(t\right) &= \exp\left[\int_0^t ds\left(s-\frac{1}{2}\right)+\int_0^t dB\left(s\right)\right] \\ &= \exp\left[\frac{1}{2}t\left(t-1\right)+B\left(t\right)\right] \end{aligned} \tag{3}$$

we compute the Itô differential of the stochastic process $U\left(t\right)=f\left(t,Y\left(t\right)\right):=\frac{1}{Y(t)}$ obtaining

$$dU\left(t\right)=\left[-t+1\right]U\left(t\right)dt-U\left(t\right)dB\left(t\right)\ .$$

Hence, the Itô differential of the product $X\left(t,X_0\right)U\left(t\right)$ is

$$d\left(X\left(t,X_0\right)U\left(t\right)\right)=2tU\left(t\right)dt+tU\left(t\right)dB\left(t\right)\ .$$

Therefore, since $X\left(0,X_0\right)U\left(0\right)=X_0$,

$$X\left(t,X_0\right)U\left(t\right)=X_0+2\int_0^t dssU\left(s\right)+\int_0^t sU\left(s\right)dB\left(s\right)$$

that is

$$\begin{aligned} X\left(t,X_0\right) &= Y\left(t\right)\left\{X_0+2\int_0^t ds\frac{s}{Y\left(s\right)}+\int_0^t\frac{s}{Y\left(s\right)}dB\left(s\right)\right\} \\ &= \exp\left[\frac{1}{2}t\left(t-1\right)+B\left(t\right)\right]\times \\ &\quad\times\left\{X_0+2\int_0^t dsse^{-\frac{2}{2}\left(s-1\right)-B\left(s\right)}+\right. \\ &\quad\left.+\int_0^t se^{-\frac{2}{2}\left(s-1\right)-B\left(s\right)}dB\left(s\right)\right\}\ . \end{aligned}$$

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**Exercise 3** *Compute the characteristic function of the random vector* $(\log Y(1,1), \log Y(2,1))$, *where* $(Y(t,1)\ ,\ t \geq 0)$ *is the stochastic process solution of the homogeneuous Itô SDE associated to the equation (2).*

**Solution:** Let $Z(t) := \log Y(t)$ where $Y(t)$ is given in (3). Then,

$$\mathbb{E}[Z(t)] = \frac{1}{2} t(t-1)$$

and

$$Cov[Z(t), Z(s)] = Cov[B(t)B(s)] = (t \wedge s)\ .$$

$(Z(1), Z(2))$ is gaussian random vector with parameters $\mu := (0,1)$ and covariance matrix

$$C := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\ .$$

Hence is characteristic function is

$$\mathbb{R}^2 \ni (\lambda_1, \lambda_2) \longmapsto \varphi_\zeta(\lambda_1, \lambda_2) = e^{i\lambda_2 - \frac{1}{2}\left(1\lambda_1^2 + 2\lambda_1\lambda_2 + 2\lambda_2^2\right)} \in \mathbb{C}\ .$$

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