

Solution of the exercises of the exam of the course
Probability and Stochastic Processes

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Exercise 1 Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be the filtered Wiener space. Consider the stochastic process described by the Itô Stochastic Differential Equation

$$\begin{aligned} X(t, X_0) &= X_0 + \int_0^t ds e^{-s} X(s) + \int_0^t 2s dB(s) , \\ dX(t) &= e^{-t} X(t) dt + 2t dB(t) . \end{aligned} \tag{1}$$

where $(B(t), t \geq 0)$ is the Brownian motion.

1. Solve the equation (1) assuming the initial datum $X_0 = 0$.
2. Compute the probability distribution of the stochastic process $(X(t, X_0), t \geq 0)$.
3. Compute the density of the random vector $(X(2, X_0), X(1, X_0))$.

Solution: The equation (1) is an Itô EDS with additive noise.

Setting

$$Y(t) = f(t, X(t)) := X(t) e^{-\int_0^t ds e^{-s}} = X(t) e^{e^{-t}-1} \tag{2}$$

and computing the Itô's differential of $Y(t)$, since

$$\partial_t f(t, x) = -e^{-t} f(t, x) , \tag{3}$$

$$\partial_x f(t, x) = e^{e^{-t}-1} , \tag{4}$$

$$\partial_x^2 f(t, x) = 0 , \tag{5}$$

we get

$$dY(t) = 2te^{e^{-t}-1} dB(t) , \tag{6}$$

$$Y(t) = \int_0^t 2se^{e^{-s}-1} dB(s) . \tag{7}$$

That is, taking into account that $Y(0, X(0)) = 0$,

$$\begin{aligned} X(t, X_0) &= e^{1-e^{-t}} \int_0^t 2se^{e^{-s}-1} dB(s) \\ &= \int_0^t 2se^{e^{-s}-e^{-t}} dB(s) . \end{aligned} \quad (8)$$

1. We have

$$\mathbb{E}[X(t, X_0)] = 0 . \quad (9)$$

Moreover

$$q_X(t) = \mathbb{E}[X^2(t)] = \int_0^t ds 4s^2 e^{2(e^{-s}-e^{-t})} . \quad (10)$$

Hence, $X(t)$ has Gaussian distribution of mean 0 and variance $q_X(t)$.

2. From (8) it follows that

$$\text{Cov}[X(t, X_0), X(s, X_0)] = e^{-(e^{-t}+e^{-s})} \mathbb{E} \left[\int_0^{t \wedge s} d\tau 4\tau^2 e^{2e^{-\tau}} \right] \quad (11)$$

The random vector

$$\begin{aligned} Y &: = (X(2, X_0), X(1, X_0)) \\ &= \left(\int_0^2 2se^{e^{-s}-e^{-2}} dB(s), \int_0^1 2se^{e^{-s}-e^{-1}} dB(s) \right) \end{aligned} \quad (12)$$

has Gaussian distribution with expectation vector $(0, 0)$ and covariance matrix

$$C := \begin{pmatrix} a & b \\ b & c \end{pmatrix} . \quad (13)$$

Therefore,

$$\begin{aligned} f_Y(x, y) &= \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp \left\{ -\frac{\langle C^{-1}(x, y), (x, y) \rangle}{2} \right\} \\ &= \frac{1}{\sqrt{(2\pi)^2 (ac - b^2)}} \exp \left\{ -\frac{1}{2(ac - b^2)} (bx^2 - 2bxy + cy^2) \right\} . \end{aligned} \quad (14)$$

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Exercise 2 Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be the filtered Wiener space. Consider the stochastic process described by the Itô Stochastic Differential Equation

$$\begin{aligned} X(t, X_0) &= X_0 + t + 2 \int_0^t \sqrt{X(s)} dB(s) , \\ \begin{cases} dX(t) = dt + 2\sqrt{X(t)} dB(t) \\ X(0) = X_0 \end{cases} . \end{aligned}$$

where $(B(t), t \geq 0)$ is the Brownian motion and X_0 is a r.v. independent of the filtration generated by the Brownian motion.

What is the probability distribution of the solution at a given time t ?

Solution: Set $X(t, X_0) = f(t, B(t))$ where $f \in C^{1,2}(\mathbb{R}^2)$. By the Itô's Lemma we get

$$df(B(t)) = \left[\left(\frac{\partial}{\partial t} f \right) (t, B(t)) + \frac{1}{2} \left(\frac{\partial^2}{\partial y^2} f \right) (t, B(t)) \right] dt + \left(\frac{\partial}{\partial y} f \right) (t, B(t)) dB(t) .$$

Hence,

$$\begin{aligned} X_0 + t + 2 \int_0^t \sqrt{X(s)} dB(s) &= X_0 + \int_0^t ds + 2 \int_0^t \sqrt{f(B(s), X_0)} dB(s) \\ &= f(0, 0) + \int_0^t ds \left[\left(\frac{\partial}{\partial t} f \right) + \frac{1}{2} \left(\frac{\partial^2}{\partial y^2} f \right) \right] (s, B(s)) + \int_0^t \left(\frac{\partial}{\partial y} f \right) (s, B(s)) dB(s) , \end{aligned}$$

which implies

$$\begin{cases} \left(\frac{\partial}{\partial y} f \right) (t, y) = 2\sqrt{f(t, y)} \\ \left(\frac{\partial}{\partial t} f \right) (t, y) + \frac{1}{2} \left(\frac{\partial^2}{\partial y^2} f \right) (t, y) = 1 \\ f(0, 0) = X_0 \end{cases} .$$

Differentiating the first equation with respect to y we get

$$\left(\frac{\partial^2}{\partial y^2} f \right) = \frac{1}{\sqrt{f(t, y)}} \left(\frac{\partial}{\partial y} f \right) (t, y) = 2$$

which, inserted in the second equation gives $\left(\frac{\partial}{\partial t} f \right) (t, y) = 0$. Hence, $f(t, y) = y^2 + X_0$, that is $X(t, X_0) = B^2(t) + X_0$. ■