Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ *be the filtered Wiener space. We consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t,X_0)=X_0+\int_0^t dsX(s)+\int_0^t sX(s)\,dB(s)\ , \tag{1}$$
$$dX(t)=X(t)\,dt+tX(t)\,dB(t)\ .$$

where $\{B(t)\}_{t\geq 0}$ *is the Brownian motion.*

1. *Solve the equation (1) assuming the initial datum* $X_0=1$.
2. *Compute the probability distribution of the r.v.* $X(t,X_0)\,,t\geq 0$.
3. *Compute the density of the random vector* $(\log X(2,X_0)\,,\log X(1,X_0))\,.$

Solution:

1. The equation (1) is an Itô EDS with moltiplicative noise.
   Setting
   $$Y(t)=f(t,X(t)):=\log\frac{X(t)}{X_0}$$
   and computing the Itô's differential of $Y(t)$, since $f(t,x)=\log x$ and
   $$\partial_t f(t,x)=0\ ,$$
   $$\partial_x f(t,x)=\frac{1}{x}\ ,$$
   $$\partial_x^2 f(t,x)=-\frac{1}{x^2}\ ,$$
   we get
   $$dY(t)=\left(1-\frac{1}{2}t^2\right)dt+tdB(t)\ ,$$
   $$Y(t)=\int_0^t ds\left(1-\frac{1}{2}s^2\right)+\int_0^t sdB(s)\ .$$

That is, taking into account that $Y(0, X(0)) = 0$,

$$X(t, X_0) = X_0 e^{t - \frac{t^3}{6} + \int_0^t s dB(s)} .$$

2. Since $X_0 = 1$, from the previous equation we have that $\mathbb{P}\{X(t) \leq x\}$, where $X(t) := X(t, 1)$ is equal to $\mathbb{P}\left\{\int_0^t s dB(s) \leq \log x + \frac{t^3}{6} - t\right\}$, but $\int_0^t s dB(s)$ is a Gaussian centered r.v. with variance $\frac{t^3}{3}$.

3. From (**??**) it follows that

$$\begin{aligned}
\mathbb{E}[Y(t, X_0)] &= t - \frac{t^3}{6} \\
Cov[Y(t, X_0), Y(s, X_0)] &= \mathbb{E}[(Y(t, X_0) - \mathbb{E}[Y(t, X_0)])(Y(s, X_0) - \mathbb{E}[Y(s, X_0)])] \\
&= \mathbb{E}\left[\int_0^{t \wedge s} d\tau \tau^2\right] = \frac{(t \wedge s)^3}{3}
\end{aligned}$$

The random vector

$$\begin{aligned}
Y \quad &: = (\log X(2, X_0), \log X(1, X_0)) \\
&= \left(2 - \frac{4}{3} + \int_0^2 t dB(t), 1 - \frac{1}{6} + \int_0^1 t dB(t)\right)
\end{aligned}$$

has Gaussian distribution with expectation vector

$$\mu = \left(\frac{2}{3}, \frac{5}{6}\right) = (\mu_1, \mu_2)$$

and covariance matrix

$$C := \begin{pmatrix} \frac{8}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} a & b \\ b & b \end{pmatrix} .$$

Therefore,

$$\begin{aligned}
f_Y(x, y) &= \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left\{-\frac{\langle C^{-1}(x - \mu_1, y - \mu_2), (x - \mu_1, y - \mu_2)\rangle}{2}\right\} && (2) \\
&= \frac{3}{2\pi\sqrt{7}} \exp\left\{-\frac{1}{2}\left[\frac{3}{7}\left(x - \frac{2}{3}\right)^2 + \frac{6}{7}\left(x - \frac{2}{3}\right)\left(y - \frac{5}{6}\right) + \frac{24}{7}\left(y - \frac{5}{6}\right)^2\right]\right\} .
\end{aligned}$$

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Exercise 2 *Let $\{f_i\}_{i \geq 1}$ be a sequence of bounded measurable functions $f_i : \mathbb{R}^i \mapsto \mathbb{R}$ and $\{X_i\}_{i \geq 1}$ a sequence of r.v.'s on a sample space $(\Omega, \mathcal{F}, \mathbb{P})$.*

1. *What is the minimal assumption on the sequence $\{X_i\}_{i \geq 1}$ for the sequence $\{Y_i\}_{i \geq 0}$ such that $Y_0 = 0$ and $\forall i \geq 1, Y_{i+1} := \sum_{j=1}^{i+1} X_j f_{j-1}(X_1, .., X_{j-1})$ to be a martingale w.r.t. the natural filtration $\{\mathcal{F}_i\}_{i \geq 0}$ generated by $\{X_i\}_{i \geq 1}$?*

2. *Prove that if* $\{Y_i\}_{i\geq 0}$ *is a martingale w.r.t.* $\{\mathcal{F}_i\}_{i\geq 0}$*, then the sequence* $\{Z_i\}_{i\geq 1}$ *such that,* $\forall i \geq 0, Z_i := e^{Y_i}$ *is a submartingale and compute its Doob-Meyer decomposition.*

Solution: For any $i \geq 1, \mathbb{E}[|Y_i|] \leq \sup_{1\leq j\leq i+1} \sup_{(x_1,..,x_j)\in\mathbb{R}^j} |f_{j-1}(x_1,..,x_j)| \sum_{j=1}^{i+1} \mathbb{E}[|X_j|]$. Moreover, since $Y_{i+1} = Y_i + X_{i+1}W_i$ where $\forall i \geq 1, W_i := f_{i-1}(X_1,..,X_{i-1})$,

$$\mathbb{E}[Y_{i+1}|\mathcal{F}_i] = \mathbb{E}[Y_i + X_{i+1}W_i|\mathcal{F}_i] = Y_i + W_i\mathbb{E}[X_{i+1}|\mathcal{F}_i] \ .$$

Hence, if, for any $i \geq 1, \mathbb{E}[|X_i|] < \infty$ and $\mathbb{E}[X_{i+1}|\mathcal{F}_i] = 0$, then $\{Y_i\}_{i\geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_i\}_{i\geq 0}$.

Moreover, since $x \mapsto e^x$ is a convex function, by the Jensen inequality

$$\mathbb{E}[Z_{i+1}|\mathcal{F}_i] = \mathbb{E}\left[e^{Y_{i+1}}|\mathcal{F}_i\right] \geq \exp\mathbb{E}[Y_{i+1}|\mathcal{F}_i] = e^{Y_i} = Z_i \ ,$$

that is $\{Z_i\}_{i\geq 1}$ is a submartingale. The Doob-Meyer decomposition of $\{Z_i\}_{i\geq 0}$ is then given by the martingale $\{m_i\}_{i\geq 0}$ where $\forall i \geq 0, m_i := Z_i - A_i$ and by the compensator $\{A_i\}_{i\geq 0}$ where $\forall i \geq 0, A_i := Z_0 + \sum_{j=1}^{i} (\mathbb{E}[Z_j|\mathcal{F}_{j-1}] - Z_{j-1})$. ■