Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ *be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t,X_0)=X_0+\int_0^t dse^{-s}X(s)+\int_0^t sdB(s)\ , \tag{1}$$
$$dX(t)=e^{-t}X(t)\,dt+tdB(t)\ .$$

where $(B(t),t\geq 0)$ *is the Brownian motion.*

1. *Compute the covariance function of the stochastic process* $(X(t,1)\ ,\ t\geq 0)$.

Solution: The equation (1) sia a Itô's SDE with additive noise.

Setting
$$Y(t)=f(t,X(t)):=X(t)\,e^{-\int_0^t dse^{-s}}=X(t)\,e^{e^{-t}-1} \tag{2}$$
and computing the Itô's differential of $Y(t)$, since
$$\partial_t f(t,x)=-e^{-t}f(t,x)\ , \tag{3}$$
$$\partial_x f(t,x)=e^{e^{-t}-1}\ , \tag{4}$$
$$\partial_x^2 f(t,x)=0\ , \tag{5}$$
taking into account that $Y(0,X(0))=Y(0,X_0)=X_0$, we get
$$dY(t)=te^{e^{-t}-1}dB(t)\ ,\ Y(t)=X_0+\int_0^t se^{e^{-s}-1}dB(s)\ . \tag{6}$$

That is,
$$X(t,X_0)=e^{1-e^{-t}}X_0+\int_0^t se^{e^{-s}-e^{-t}}dB(s)\ . \tag{7}$$

Setting $X_0=1$, from (1) we get
$$\mathbb{E}[X(t,1)]=e^{1-e^{-t}}\ , \tag{8}$$

Hence, the covariance function of $(X(t,1)\ ,\ t\geq 0)$ is

$$\begin{aligned}
&\mathbb{E}\left[\left(X(t,1)-\mathbb{E}X(t,1)\right)\left(X(s,1)-\mathbb{E}\left[X(s,1)\right]\right)\right]= \\
&=\mathbb{E}\left[\int_0^t \tau e^{e^{-\tau}-e^{-t}}dB(\tau)\int_0^s \tau e^{e^{-\tau}-e^{-s}}dB(\tau)\right] \\
&=\mathbb{E}\left[e^{-\left(e^{-t}+e^{-s}\right)}\int_0^t \tau e^{e^{-\tau}}dB(\tau)\int_0^s \tau e^{e^{-\tau}}dB(\tau)\right] \\
&=e^{e^{-t}+e^{-s}}\int_0^{t\wedge s}d\tau\tau^2 e^{2e^{-\tau}}
\end{aligned} \tag{9}$$

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Exercise 2 *Compute the characteristic function of the random vector* $(X(1,1), X(2,1))$, *where* $(X(t,1)\ ,\ t\geq 0)$ *is the stochastic process solution of the homogeneuous Itô SDE associated to the equation (1).with initial datum* $X_0 = 1$.

Solution: The distribution of the random vector

$$\begin{aligned}
Y\quad &:= (X(1,1), X(2,1)) \\
&= \left(e^{1-e^{-1}}+\int_0^1 s e^{e^{-s}-e^{-1}}dB(s), e^{e^{-2}-1}+\int_0^2 s e^{e^{-s}-e^{-2}}dB(s)\right)
\end{aligned} \tag{10}$$

is Gaussian with parameters

$$\mu=(\mu_1,\mu_2)=\left(e^{1-e^{-1}},e^{1-e^{-2}}\right) \tag{11}$$

and, covariance matrix

$$C:=\begin{pmatrix} a & b \\ b & c \end{pmatrix}\ , \tag{12}$$

where

$$a:=e^{2e^{-1}}\int_0^1 d\tau\tau^2 e^{2e^{-\tau}}\ ;\ b:=e^{-\left(e^{-1}+e^{-2}\right)}\int_0^2 d\tau\tau^2 e^{2e^{-\tau}}\ ;\ c:=e^{2e^{-2}}\int_0^2 d\tau\tau^2 e^{2e^{-\tau}} \tag{13}$$

Thus, $\forall\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2$,

$$\begin{aligned}
\varphi_Y(\lambda)\quad &:= \mathbb{E}\left[e^{i\langle\lambda,Y\rangle}\right]=e^{i\langle\lambda,\mu\rangle-\frac{1}{2}\langle C\lambda,\lambda\rangle} \\
&= \exp\left[i(\lambda_1\mu_1+\lambda_2\mu_2)-\frac{1}{2}\left(a\lambda_1^2+2b\lambda_1\lambda_2+c\lambda_2^2\right)\right]\ .
\end{aligned} \tag{14}$$

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