Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. We consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 - \int_0^t sX(s)\, dB(s) \ , \tag{1}$$
$$dX(t) = -tX(t)\, dB(t) \ .$$

where $\{B(t)\}_{t\geq 0}$ *is the Brownian motion.*

1. *Solve the equation (1) assuming the initial datum* $X_0 = 1$.
2. *Compute the variance of* $(X(t, X_0) \ , \ t \geq 0)$.
3. *Compute the density of the random vector* $(\log X(2, X_0), \log X(1, X_0))$.

Solution:

1. The equation (1) is an Itô EDS with moltiplicative noise.
   Setting
   $$Y(t) = f(t, X(t)) := \log \frac{X(t)}{X_0} \tag{2}$$
   and computing the Itô's differential of $Y(t)$, since $f(t,x) = \log x$ and
   $$\partial_t f(t,x) = 0 \ , \tag{3}$$
   $$\partial_x f(t,x) = \frac{1}{x} \ , \tag{4}$$
   $$\partial_x^2 f(t,x) = -\frac{1}{x^2} \ , \tag{5}$$
   we get
   $$dY(t) = -\frac{1}{2}t^2 dt - t dB(t) \ , \tag{6}$$
   $$Y(t) = -\frac{1}{2}\int_0^t ds s^2 - \int_0^t s dB(s) \ . \tag{7}$$

That is, taking into account that $Y(0, X(0)) = 0$,

$$X(t, X_0) = X_0 e^{-\frac{1}{2}\int_0^t dss^2 - \int_0^t sdB(s)} \tag{8}$$
$$= X_0 e^{-\frac{t^3}{6} - \int_0^t sdB(s)} \ .$$

2. Since $X_0 = 1$, from (1) we have

$$\mathbb{E}[X(t, X_0)] = 1 - \mathbb{E}\int_0^t sX(s)\, dB(s) = 1 \ . \tag{9}$$

Moreover, setting $Y(t) := f(t, X(t)) = X^2(t)$ and computing its Itô's differential we obtain

$$\begin{aligned} Y(t) &= 1 + \int_0^t dss^2 X^2(s) - \int_0^t 2X^2(s)\, sdB(s) \\ &= 1 + \int_0^t dss^2 Y(s) - \int_0^t 2Y(s)\, sdB(s) \ . \end{aligned} \tag{10}$$

Hence, setting $q_X(t) := \mathbb{E}[Y(t)] = \mathbb{E}\left[X^2(t)\right]$ and taking the expectation on both sides of the previous expression, differentiating w.r.t. t we get that q_X is the solution of the Chauchy problem

$$\begin{cases} \frac{d}{dt} q_X(t) = t^2 q_X(t) \\ q_X(0) = 1 \end{cases} \tag{11}$$

that is,

$$q_X(t) = e^{\frac{t^3}{3}} \ . \tag{12}$$

The variance of $X(t, X_0)$ is then

$$\mathbb{E}\left[X^2(t, X_0)\right] - \mathbb{E}^2[X(t, X_0)] = q_X(t) - 1 = e^{\frac{t^3}{3}} - 1 \ . \tag{13}$$

3. From (7) it follows that

$$\mathbb{E}[Y(t, X_0)] = -\frac{t^3}{6} \tag{14}$$

$$\begin{aligned} Cov[Y(t, X_0), Y(s, X_0)] &= \mathbb{E}[(Y(t, X_0) - \mathbb{E}[Y(t, X_0)])(Y(s, X_0) - \mathbb{E}[Y(s, X_0)])] \\ &= \mathbb{E}\left[\int_0^{t\wedge s} d\tau\tau^2\right] = \frac{(t\wedge s)^3}{3} \end{aligned} \tag{15}$$

The random vector

$$\begin{aligned} Y \ :&= (\log X(2, X_0), \log X(1, X_0)) \\ &= \left(-\frac{4}{3} - \int_0^2 tdB(t), -\frac{1}{6} + \int_0^1 tdB(t)\right) \end{aligned} \tag{16}$$

has Gaussian distribution with expectation vector

$$\mu = \left(-\frac{4}{3}, -\frac{1}{6}\right) = (\mu_1, \mu_2) \tag{17}$$

and covariance matrix

$$C := \begin{pmatrix} \frac{8}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} a & b \\ b & b \end{pmatrix} . \tag{18}$$

Therefore,

$$\begin{aligned} f_Y(x,y) &= \frac{1}{\sqrt{(2\pi)^2 \det C^{-1}}} \exp\left\{ -\frac{\left\langle (x-\mu_1, y-\mu_2), C^{-1}(x-\mu_1, y-\mu_2) \right\rangle}{2} \right\} \\ &= \frac{\sqrt{7}}{6\pi} \exp\left\{ -\frac{1}{2}\left[\frac{3}{7}\left(x+\frac{4}{3}\right)^2 + \frac{6}{7}\left(x+\frac{4}{3}\right)\left(y+\frac{1}{6}\right) + \frac{24}{7}\left(y+\frac{1}{6}\right)^2 \right] \right\} . \end{aligned} \tag{19}$$

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Exercise 2 *An urn contains black and white balls. A ball is drawn at random from the urn and it is returned together with 2 more balls of the same colour. Let B_n be the number of black balls and W_n the number of white balls at stage n. If at the beginning the urn contains a black and a white ball, i.e. $B_0 = W_0 = 1$, prove that the sequence $\{X_n\}_{n\geq 0}$ such that $X_n := \frac{B_n}{2n+2}$ is a martingale w.r.t. its natural filtration and study its convergence.*

Solution: Let P_n be the total number of balls in the urn at the time of the $n+1$ extraction; we have

$$P_n = W_n + B_n = 2n + 2 . \tag{20}$$

Moreover, since we sample with replacement, the probability of drawing a black ball from the urn at the n-th extraction is $\frac{B_n}{P_n}$. Hence, the probability that the number of black balls after the n-th extraction stays the same is the probability of drawing a white ball, namely

$$\mathbb{P}[B_{n+1} = m | B_n = m] = 1 - \frac{m}{2n+2} . \tag{21}$$

Therefore,

$$\mathbb{P}[B_{n+1} = m+2 | B_n = m] = \frac{m}{2n+2} . \tag{22}$$

Let $\{\mathcal{F}_n\}_{n\geq 0}$ be the filtration generated by the sequence of r.v.'s $\{X_n\}_{n\geq 0}$, i.e. such that $\mathcal{F}_n := \sigma\{X_1, .., X_n\}$ and $\mathcal{F}_0$ is the trivial σalgebra, since $\forall n \geq 1, \mathcal{F}_n = \sigma\{N_1, .., N_n\}$ we have

$$\begin{aligned} \mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\frac{B_{n+1}}{2(n+1)+2} \middle| \mathcal{F}_n \right] = \mathbb{E}\left[\frac{B_{n+1}}{2(n+1)+2} \middle| B_n \right] \\ &= \frac{B_n}{2(n+1)+2}\left(1 - \frac{B_n}{2n+2}\right) + \frac{B_n+2}{2(n+1)+2}\frac{B_n}{2n+2} \\ &= \frac{B_n}{2(n+1)+2} + \frac{2}{2(n+1)+2}X_n \\ &= X_n\left(\frac{1}{1+\frac{2}{2n+2}}\right) + \frac{2}{2(n+1)+2}X_n \\ &= X_n\left(\frac{2n+2}{2(n+1)+2}\right) + \frac{2}{2(n+1)+2}X_n = X_n . \end{aligned} \tag{23}$$

Hence $\{X_n\}_{n\geq 0}$ is a non-negative martingale and, since $\mathbb{E}[X_n] = \mathbb{E}[X_0] = \frac{1}{2}$,

$$\sup_{n\geq 0} \mathbb{E}[|X_n|] = \sup_{n\geq 0} \mathbb{E}[X_n] = \mathbb{E}[X_0] < \infty \tag{24}$$

which implies that it is also a L^1-martingale and therefore convergent $\mathbb{P}$-a.s. to a non-negative r.v. X such that $\mathbb{E}[X] = \mathbb{E}[X_0] = \frac{1}{2}$. Moreover, since by definition $B_n \geq B_0 = 1$, from (20) we get $N_n \leq 2n+1$, thus

$$\sup_{n\geq 0} \mathbb{E}\left[X_n^2\right] = \sup_{n\geq 0} \mathbb{E}\left[\frac{N_n^2}{(2n+2)^2}\right] \leq \sup_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^2 \leq 1 \ , \tag{25}$$

that is $\{X_n\}_{n\geq 0}$ is a L^2-martingale and consequently a regular martingale. ■