## Solution of the execises of the exam of the course Probability and Stochastic Processes

## $\substack{\text{a.y.}2023/2024\\02/23/2024}$

**Exercise 1** Let  $\{\xi_n\}_{n\geq 0}$  be a sequence or *.v.*'s such that  $\xi_0 \stackrel{d}{=} \exp(1)$  and  $\forall n \geq 1, \xi_n$  has a exponential distribution of parameter 1 supported on  $(\xi_{n-1}, +\infty)$ . Prove that the sequence of *r.v.*'s  $\{\eta_n\}_{n\geq 0}$  such that  $\forall n \geq 0, \eta_n := 2^n e^{-\xi_n}$  is a martingale w.r.t. the filtration generated by the sequence of *r.v.*'s  $\{\xi_n\}_{n\geq 0}$ . Is it a convergent martingale?

Solution:  $\forall n \geq 0$ ,

$$\mathbb{E}\left[|\eta_n|\right] = \mathbb{E}\left[\eta_n\right] \le 2^n < \infty$$

Moreover, denoting by  $\mathcal{F}_n$  the  $\sigma$ algebra generated by  $(\xi_1, .., \xi_n)$ ,

$$\begin{split} \mathbb{E} \left[ \eta_{n+1} | \mathcal{F}_n \right] &= 2^{n+1} \mathbb{E} \left[ e^{-\xi_{n+1}} | \mathcal{F}_n \right] \\ &= 2^{n+1} \mathbb{E} \left[ e^{-\xi_{n+1}} | \xi_n \right] \\ &= 2^{n+1} e^{\xi_n} \int_{\xi_n}^{\infty} dx e^{-2x} \\ &= 2^{n+1} \frac{e^{\xi_n}}{2} \int_{2\xi_n}^{\infty} dy e^{-y} = \eta_n \, \mathbb{P} - a.s.. \end{split}$$

Since  $\{\eta_n\}_{n\geq 0}$  is a positive martingale, it is convergent and  $\mathbb{E}[\eta_n] = \mathbb{E}[\eta_0] = \mathbb{E}[e^{-\xi_0}] = \frac{1}{2}$ .

**Exercise 2** Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$  be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation

$$X(t, X_0) = X_0 + \int_0^t dss \left( X(s) + 1 \right) + \int_0^t s \left( X(s) + 1 \right) dB(s) , \qquad (1)$$
  
$$dX(t) = t \left( X(t) + 1 \right) dt + t \left( X(t) + 1 \right) dB(t) .$$

where  $\{B(t)\}_{t>0}$  is the Brownian motion.

**Solution:** (1) is a linear Itô SDE. Considering the associated homogeneous equation with initial datum equal to 1,

$$Y(t) = 1 + \int_0^t ds s X(s) + \int_0^t s X(s) \, dB(s) \, ,$$

whose solution is

$$Y(t) = \exp\left[\int_{0}^{t} ds \left(s - \frac{s^{2}}{2}\right) + \int_{0}^{t} s dB(s)\right]$$
  
=  $\exp\left[\frac{t^{2}}{2} - \frac{t^{3}}{6} + \int_{0}^{t} s dB(s)\right]$  (2)

we compute the Itô differential of the stochastic process  $U(t) = f(t, Y(t)) := \frac{1}{Y(t)}$  obtaining

$$dU(t) = [-t + t^{2}] U(t) dt - tU(t) dB(t)$$
.

Hence, the Itô differential of the product  $X(t, X_0) U(t)$  is

$$d(X(t, X_0) U(t)) = [t + t^2] U(t) dt + tU(t) dB(t)$$

Therefore, since  $X(0, X_0) U(0) = X_0$ ,

$$X(t, X_0) U(t) = X_0 + \int_0^t ds \left[s + s^2\right] U(s) + \int_0^t s U(s) dB(s)$$

that is

$$\begin{aligned} X\left(t,X_{0}\right) &= Y\left(t\right)\left\{X_{0}+\int_{0}^{t}ds\frac{s+s^{2}}{Y\left(s\right)}+\int_{0}^{t}\frac{s}{Y\left(s\right)}dB\left(s\right)\right\} \\ &= \exp\left[\frac{t^{2}}{2}\left(1-\frac{t}{3}\right)+\int_{0}^{t}sdB\left(s\right)\right]\times \\ &\times\left\{X_{0}+\int_{0}^{t}ds\left(s+s^{2}\right)e^{-\frac{s^{2}}{2}\left(1-\frac{s}{3}\right)-\int_{0}^{s}\tau dB(\tau)}+\right. \\ &\left.+\int_{0}^{t}se^{-\frac{s^{2}}{2}\left(1-\frac{s}{3}\right)-\int_{0}^{s}\tau dB(\tau)}dB\left(s\right)\right\} \end{aligned}$$

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**Exercise 3** Compute the characteristic function of the random vector  $(\log Y(1,1), \log Y(2,1))$ , where  $(Y(t,1), t \ge 0)$  is the stochastic process solution of the homogeneuous Itô SDE associated to the equation (1).

**Solution:** Let  $Z(t) := \log Y(t)$  where Y(t) is given in (2). Then,

$$\mathbb{E}\left[Z\left(t\right)\right] = \frac{t^2}{2} - \frac{t^3}{6}$$

and

$$\begin{split} Cov\left[Z\left(t\right), Z\left(s\right)\right] &= & \mathbb{E}\left[\int_{0}^{t} \tau dB\left(\tau\right) \int_{0}^{s} \tau dB\left(\tau\right)\right] \\ &= & \int_{0}^{t \wedge s} d\tau \tau^{2} = \frac{\left(t \wedge s\right)^{3}}{3} \; . \end{split}$$

(Z(1), Z(2)) is gaussian random vector with parameters  $\mu := (\frac{1}{3}, \frac{2}{3})$  and covariance matrix

$$C:= \left(\begin{array}{cc} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{8}{3} \end{array}\right) \ .$$

Hence is characteristic function is

$$\mathbb{R}^2 \ni (\lambda_1, \lambda_2) \longmapsto \varphi_{\zeta} \left( \lambda_1, \lambda_2 \right) = e^{i \left( \frac{1}{3} \lambda_1 + \frac{2}{3} \lambda_2 \right) - \frac{1}{2} \left( \frac{1}{3} \lambda_1^2 + \frac{2}{3} \lambda_1 \lambda_2 + \frac{8}{3} \lambda_2^2 \right)} \in \mathbb{C} \ .$$