Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let* $\{\xi_n\}_{n\geq 0}$ *be a sequence o r.v.'s such that* $\xi_0 \stackrel{d}{=} \exp(1)$ *and* $\forall n \geq 1, \xi_n$ *has a exponential distribution of parameter* 1 *supported on* $(\xi_{n-1}, +\infty)$*. Prove that the sequence of r.v.'s* $\{\eta_n\}_{n\geq 0}$ *such that* $\forall n \geq 0, \eta_n := 2^n e^{-\xi_n}$ *is a martingale w.r.t. the filtration generated by the sequence of r.v.'s* $\{\xi_n\}_{n\geq 0}$*. Is it a convergent martingale?*

Solution: $\forall n \geq 0$,

$$\mathbb{E}[|\eta_n|] = \mathbb{E}[\eta_n] \leq 2^n < \infty \ ,$$

Moreover, denoting by $\mathcal{F}_n$ the σalgebra generated by $(\xi_1, .., \xi_n)$,

$$\begin{aligned}
\mathbb{E}\left[\eta_{n+1}|\mathcal{F}_n\right] &= 2^{n+1}\mathbb{E}\left[e^{-\xi_{n+1}}|\mathcal{F}_n\right] \\
&= 2^{n+1}\mathbb{E}\left[e^{-\xi_{n+1}}|\xi_n\right] \\
&= 2^{n+1}e^{\xi_n}\int_{\xi_n}^{\infty} dx e^{-2x} \\
&= 2^{n+1}\frac{e^{\xi_n}}{2}\int_{2\xi_n}^{\infty} dy e^{-y} = \eta_n \ \mathbb{P}-a.s..
\end{aligned}$$

Since $\{\eta_n\}_{n\geq 0}$ is a positive martingale, it is convergent and $\mathbb{E}[\eta_n] = \mathbb{E}[\eta_0] = \mathbb{E}\left[e^{-\xi_0}\right] = \frac{1}{2}$. ■

Exercise 2 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 + \int_0^t dss(X(s)+1) + \int_0^t s(X(s)+1)\,dB(s) \ , \tag{1}$$
$$dX(t) = t(X(t)+1)\,dt + t(X(t)+1)\,dB(t) \ .$$

where $\{B(t)\}_{t\geq 0}$ *is the Brownian motion.*

Solution: (1) is a linear Itô SDE. Considering the associated homogeneous equation with initial datum equal to 1,

$$Y(t) = 1 + \int_0^t dssX(s) + \int_0^t sX(s)\,dB(s) \ ,$$

whose solution is

$$
\begin{aligned}
Y(t) &= \exp\left[\int_0^t ds\left(s-\frac{s^2}{2}\right)+\int_0^t s dB(s)\right] \qquad (2)\\
&= \exp\left[\frac{t^2}{2}-\frac{t^3}{6}+\int_0^t s dB(s)\right]
\end{aligned}
$$

we compute the Itô differential of the stochastic process $U(t)=f(t,Y(t)):=\frac{1}{Y(t)}$ obtaining

$$dU(t)=\left[-t+t^2\right]U(t)\,dt-tU(t)\,dB(t) \ .$$

Hence, the Itô differential of the product $X(t,X_0)\,U(t)$ is

$$d\left(X(t,X_0)\,U(t)\right)=\left[t+t^2\right]U(t)\,dt+tU(t)\,dB(t) \ .$$

Therefore, since $X(0,X_0)\,U(0)=X_0$,

$$X(t,X_0)\,U(t)=X_0+\int_0^t ds\left[s+s^2\right]U(s)+\int_0^t sU(s)\,dB(s)$$

that is

$$
\begin{aligned}
X(t,X_0) &= Y(t)\left\{X_0+\int_0^t ds\frac{s+s^2}{Y(s)}+\int_0^t\frac{s}{Y(s)}dB(s)\right\}\\
&= \exp\left[\frac{t^2}{2}\left(1-\frac{t}{3}\right)+\int_0^t s dB(s)\right]\times\\
&\quad\times\left\{X_0+\int_0^t ds\left(s+s^2\right)e^{-\frac{s^2}{2}\left(1-\frac{s}{3}\right)-\int_0^s\tau dB(\tau)}+\right.\\
&\quad\left.+\int_0^t s e^{-\frac{s^2}{2}\left(1-\frac{s}{3}\right)-\int_0^s\tau dB(\tau)}dB(s)\right\} \ .
\end{aligned}
$$

■

Exercise 3 *Compute the characteristic function of the random vector* $(\log Y(1,1),\log Y(2,1))$, *where* $(Y(t,1)\ ,\ t\geq 0)$ *is the stochastic process solution of the homogeneuous Itô SDE associated to the equation (1).*

Solution: Let $Z(t):=\log Y(t)$ where $Y(t)$ is given in (2). Then,

$$\mathbb{E}\left[Z(t)\right]=\frac{t^2}{2}-\frac{t^3}{6}$$

and

$$
\begin{aligned}
Cov\left[Z(t),Z(s)\right] &= \mathbb{E}\left[\int_0^t\tau dB(\tau)\int_0^s\tau dB(\tau)\right]\\
&= \int_0^{t\wedge s}d\tau\tau^2=\frac{(t\wedge s)^3}{3} \ .
\end{aligned}
$$

$(Z(1), Z(2))$ is gaussian random vector with parameters $\mu := \left(\frac{1}{3}, \frac{2}{3}\right)$ and covariance matrix

$$C := \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{8}{3} \end{pmatrix} .$$

Hence is characteristic function is

$$\mathbb{R}^2 \ni (\lambda_1, \lambda_2) \longmapsto \varphi_\zeta (\lambda_1, \lambda_2) = e^{i\left(\frac{1}{3}\lambda_1 + \frac{2}{3}\lambda_2\right) - \frac{1}{2}\left(\frac{1}{3}\lambda_1^2 + \frac{2}{3}\lambda_1^2\lambda_2^2 + \frac{8}{3}\lambda_2^2\right)} \in \mathbb{C} .$$

■