Solution of the execises of the exam of the course *Probability and Stochastic Processes*

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Exercise 1 *Let $\{X_i\}_{i\geq 1}$ a sequence of i.i.d.r.v.'s on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{R} \ni x \longmapsto f_{X_1}(x) = \frac{e^{-|x|}}{2} \in \mathbb{R}_+$. Let $\{\mathcal{F}_n\}_{n\geq 0}$ be the natural filtration i.e. the sequence of σalgebras s.t. $\mathcal{F}_0$ is the trivial σalgebra and, for any $n \geq 1$, let $\mathcal{F}_n$ be the σalgebra generated by the random vector $(X_1, .., X_n)$. Consider the sequence $\{M_n(\theta)\}_{n\geq 0}$ s.t. $M_0(\theta) := 1$ and $\forall n \geq 1, M_n(\theta) := \exp\left[\theta \sum_{k=1}^{n} X_k - nA(\theta)\right]$, where $\mathbb{R} \ni \theta \longmapsto A(\theta) \in \mathbb{R}$. such that is a martingale*

1. *For which values of $\theta, \{M_n(\theta)\}_{n\geq 0} \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$?*
2. *Compute the function $\theta \mapsto A(\theta)$ such that $\{M_n(\theta)\}_{n\geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$.*
3. *For which values of $\theta, \{M_n(\theta)\}_{n\geq 0}$ is a L^2-martingale?*

Solution:

1. For any $n \geq 0, M_n(\theta) > 0$. Moreover, given $n \geq 1$, since the X_k's are i.i.d.,

$$\begin{aligned} \mathbb{E}[M_n(\theta)] &= e^{-nA(\theta)}\mathbb{E}\left[e^{\theta \sum_{k=1}^{n} X_k}\right] = e^{-nA(\theta)}\mathbb{E}\left[\prod_{k=1}^{n} e^{\theta X_k}\right] \\ &= e^{-nA(\theta)}\mathbb{E}^n\left[e^{\theta X_1}\right] = \frac{e^{-nA(\theta)}}{2}\left(\int_{\mathbb{R}} dx e^{\theta x - |x|}\right)^n . \end{aligned}$$

   Since the last integral is finite for $|\theta| \leq 1$, we get that $M_n(\theta) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for these values of θ.

2. We have to find $(-1, 1) \ni \theta \longmapsto A(\theta) \in \mathbb{R}$ such that $\mathbb{E}[M_{n+1}(\theta) | \mathcal{F}_n] = M_n(\theta)$. Since $M_{n+1}(\theta) = M_n(\theta) \exp(\theta X_{n+1} - A(\theta))$, recalling that the X_k's are i.i.d.,

$$\mathbb{E}[M_{n+1}(\theta) | \mathcal{F}_n] = M_n(\theta)\,\mathbb{E}\left[e^{\theta X_1}\right] e^{-A(\theta)} \ ,$$

that is

$$\begin{aligned} A(\theta) &= \log \mathbb{E}\left[e^{\theta X_1}\right] = \log \frac{1}{2}\int_{\mathbb{R}} dx e^{\theta x - |x|} \\ &= \log \frac{1}{2}\left[\int_{-\infty}^{0} dx e^{x(1+\theta)} + \int_{0}^{\infty} dx e^{-x(1-\theta)}\right] \\ &= \log \frac{1}{2}\left[\frac{1}{1+\theta} + \frac{1}{1-\theta}\right] = \log \frac{1}{2\left(1-\theta^2\right)} \end{aligned}$$

3. Working as in pont 1. since $M_0^2(\theta) = M_0(\theta) = 1, \forall n \geq 1$ we get

$$\mathbb{E}\left[M_n^2(\theta)\right] = \frac{e^{-n2A(\theta)}}{2}\left(\int_{\mathbb{R}} dx e^{2\theta x - |x|}\right)^n \quad .$$

Hence $M_n(\theta) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for any $n \geq 0$, for $|\theta| \leq \frac{1}{2}$.

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Exercise 2 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$\begin{gathered} X(t, X_0) = X_0 + \int_0^t ds s\left(X(s) + 1\right) + \int_0^t s\left(X(s) + 1\right) dB(s) \quad , \\ dX(t) = t\left(X(t) + 1\right) dt + t\left(X(t) + 1\right) dB(t) \quad . \end{gathered} \tag{1}$$

where $(B(t), t \geq 0)$ *is the Brownian motion.*

1. *Compute the variance of the stochastic process* $(X(t, 1) \; , \; t \geq 0)$.

Solution: The equation (1) sia a linear Itô's SDE with multiplicative noise. Hence, Considering the associated homogeneous equation with initial datum equal to 1,

$$Y(t) = 1 + \int_0^t ds s Y(s) + \int_0^t s Y(s) dB(s) \quad ,$$

whose solution is

$$\begin{aligned} Y(t) &= \exp\left[\int_0^t ds\left(s - \frac{s^2}{2}\right) + \int_0^t s dB(s)\right] \\ &= \exp\left[\frac{t^2}{2} - \frac{t^3}{6} + \int_0^t s dB(s)\right] \quad , \end{aligned} \tag{2}$$

and computing the Itô differential of the process $U(t) = f(t, Y(t)) := \frac{1}{Y(t)}$ we get

$$dU(t) = \left[-t + t^2\right] U(t) dt - tU(t) dB(t) \quad .$$

Computing the Itô differential of the process $X(t, X_0) U(t)$ we obtain

$$d\left(X(t, X_0) U(t)\right) = \left[t - t^2\right] U(t) dt + tU(t) dB(t) \quad .$$

Therefore, since $X(0,X_0)U(0)=X_0$,

$$X(t,X_0)U(t)=X_0+\int_0^t ds\left[s-s^2\right]U(s)+\int_0^t sU(s)\,dB(s)\ ,$$

that is

$$\begin{aligned}X(t,X_0) &= Y(t)\left\{X_0+\int_0^t ds\frac{s-s^2}{Y(s)}+\int_0^t\frac{s}{Y(s)}dB(s)\right\} \quad (3)\\ &= \exp\left[\frac{t^2}{2}\left(1-\frac{t}{3}\right)+\int_0^t sdB(s)\right]\times\\ &\quad\times\left\{X_0+\int_0^t ds\left(s-s^2\right)e^{-\frac{s^2}{2}\left(1-\frac{s}{3}\right)-\int_0^s\tau dB(\tau)}+\right.\\ &\quad\left.+\int_0^t se^{-\frac{s^2}{2}\left(1-\frac{s}{3}\right)-\int_0^s\tau dB(\tau)}dB(s)\right\}\ .\end{aligned}$$

A simpler way to solve (1) would have been to consider that, adding 1 to both members of the equation we would have had

$$(X(t,X_0)+1)=(X_0+1)+\int_0^t dss(X(s)+1)+\int_0^t s(X(s)+1)\,dB(s)\ .$$

Therefore, setting $Z(t,Z_0):=X(t,X_0)+1$, with $Z_0:=X_0+1$, solving (1) is equivalent to solve

$$Z(t,Y_0)=Z_0+\int_0^t dssZ(s)+\int_0^t sZ(s)\,dB(s) \quad (4)$$

which is a homogeneous Itô SDE, which solution is given by Z_0 times (2), namely $Z(t,Z_0)=Z_0Y(t)$. Hence

$$\begin{aligned}X(t,X_0) &= (X_0+1)Y(t)-1 \quad (5)\\ &= (X_0+1)\exp\left[\frac{t^2}{2}\left(1-\frac{t}{3}\right)+\int_0^t sdB(s)\right]-1\ .\end{aligned}$$

Why (3) and (5) are two equivalent representations of the solution of (1)?

1. Taking the expectation value of both members of (1) and setting $\mu_X(t):=\mathbb{E}[X(t,1)]$, we get $\mu_X(t)=1+\int_0^t dss(\mu_X(s)+1)$ which solves the Cauchy problem

$$\begin{cases}\frac{d}{dt}\mu_X(t)=t(\mu_X(t)+1)\\ \mu(0)=1\end{cases}$$

   that is $\mu_X(t)=2e^{\frac{t^2}{2}}-1$.

   Computing the Itô differential of $Y(t):=f(t,X(t,1))=X^2(t)$ we get

$$\begin{aligned}dY(t) &= 1+\int_0^t ds\left[2sX(s)(X(s)+1)+s(X(s)+1)^2\right] \quad (6)\\ &\quad+\int_0^t 2sX(s)(X(s)+1)\,dB(s)\ .\end{aligned}$$

Taking the expectation value of both sides of the previous equation and differentiating w.r.t. t, setting $q_X(t) := \mathbb{E}[Y(t)] = \mathbb{E}\left[X^2(t,1)\right]$, we obtain tath q_X solves the Chauchy problem

$$\begin{cases} \frac{d}{dt} q_X(t) = \left(2s+s^2\right) q_X(t) + 2\left(s+s^2\right) \mu_X(t) + s^2 \\ q_X(0) = 1 \end{cases} .$$

that is

$$\begin{aligned} q_X(t) &= 1 + \int_0^t ds \left[2\left(s+s^2\right)\left(2e^{\frac{s^2}{2}} - 1\right) + s^2\right] e^{\int_s^t d\tau\left(2\tau+\tau^2\right)} \\ &= 1 + \int_0^t ds \left[2\left(s+s^2\right)\left(2e^{\frac{s^2}{2}} - 1\right) + s^2\right] e^{\left(t^2+\frac{t^3}{3}\right)-\left(s^2+\frac{s^3}{3}\right)} \end{aligned}$$

Hence,

$$Var[X(t,1)] = q_X(t) - \mu_X^2(t) .$$

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Exercise 3 *Compute the probability density function of the random variable* $\sqrt{|\log Y(1,1)|}$, *where* $(Y(t,1)\ ,\ t \geq 0)$ *is the stochastic process solution of the homogeneous Itô SDE associated to the equation (1).*

Solution: The homogeneous equation associated to (1) with initial datum equal to 1 is

$$Y(t) = 1 + \int_0^t ds s Y(s) + \int_0^t s Y(s) dB(s) ,$$

whose solution is

$$\begin{aligned} Y(t) &= \exp\left[\int_0^t ds\left(s - \frac{s^2}{2}\right) + \int_0^t s dB(s)\right] \\ &= \exp\left[\frac{t^2}{2} - \frac{t^3}{6} + \int_0^t s dB(s)\right] . \end{aligned}$$

Therefore, $Z := \log Y(1,1) = \frac{1}{3} - \int_0^1 s dB(s)$ is a Gaussian r.v.; more precisely $Z \overset{d}{=} N\left(\frac{1}{3}, \frac{2}{9}\right)$. Therefore, setting $W := \sqrt{|Z|}$,

$$\begin{aligned} F_W(w) &:= \mathbb{P}\{W \leq w\} = \mathbb{P}\left\{\sqrt{|Z|} \leq w\right\} = \mathbb{P}\left\{|Z| \leq w^2\right\} \\ &= \mathbb{P}\left\{-w^2 \leq Z \leq w^2\right\} = \int_{w^2}^{w^2} dz \frac{3e^{-\frac{9\left(z-\frac{1}{3}\right)^2}{4}}}{2\sqrt{\pi}} . \end{aligned}$$

Hence,

$$f_W(w) := \frac{d}{dw} F_W(w) = \frac{3e^{-\frac{9\left(w^2-\frac{1}{3}\right)^2}{4}}}{\sqrt{\pi}} w + \frac{3e^{-\frac{9\left(w^2+\frac{1}{3}\right)^2}{4}}}{\sqrt{\pi}} w .$$

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