Solution of the execises of the exam of the course *Probability and Stochastic Processes*

a.y. 2022/2023
06/29/2023

Exercise 1 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. Consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 + \int_0^t ds e^{-s} X(s) + \int_0^t s dB(s) \ , \qquad (1)$$
$$dX(t) = e^{-t} X(t) dt + t dB(t) \ .$$

where $(B(t), t \geq 0)$ *is the Brownian motion.*

1. *Solve the equation (1) assuming the initial datum* $X_0 = 0$.
2. *Compute the probability distribution of the stochastic process* $(X(t, X_0) \ , \ t \geq 0)$.
3. *Compute the density of the random vector* $(X(2, X_0), X(1, X_0))$.

Solution: The equation (1) is an Itô EDS with additive noise.

Setting

$$Y(t) = f(t, X(t)) := X(t) e^{-\int_0^t ds e^{-s}} = X(t) e^{e^{-t}-1} \qquad (2)$$

and computing the Itô's differential of $Y(t)$, since

$$\partial_t f(t, x) = -e^{-t} f(t, x) \ , \qquad (3)$$
$$\partial_x f(t, x) = e^{e^{-t}-1} \ , \qquad (4)$$
$$\partial_x^2 f(t, x) = 0 \ , \qquad (5)$$

we get

$$dY(t) = t e^{e^{-t}-1} dB(t) \ , \qquad (6)$$
$$Y(t) = \int_0^t s e^{e^{-s}-1} dB(s) \ . \qquad (7)$$

That is, taking into account that $Y(0, X(0)) = 0$,

$$X(t, X_0) = e^{1-e^{-t}} \int_0^t s e^{e^{-s}-1} dB(s) \tag{8}$$
$$= \int_0^t s e^{e^{-s}-e^{-t}} dB(s) \ .$$

1. We have
$$\mathbb{E}[X(t, X_0)] = 0 \ . \tag{9}$$
Moreover
$$q_X(t) = \mathbb{E}\left[X^2(t)\right] = \int_0^t ds s^2 e^{2\left(e^{-s}-e^{-t}\right)} \ . \tag{10}$$
Hence, $X(t)$ has Gaussian distribution of mean 0 and variance $q_X(t)$.

2. From (8) it follows that
$$Cov[X(t, X_0), X(s, X_0)] = e^{-\left(e^{-t}+e^{-s}\right)} \mathbb{E}\left[\int_0^{t \wedge s} d\tau \tau^2 e^{2e^{-\tau}}\right] \tag{11}$$
The random vector
$$\begin{aligned} Y \quad : &= (X(2, X_0), X(1, X_0)) \\ &= \left(\int_0^2 s e^{e^{-s}-e^{-2}} dB(s), \int_0^1 s e^{e^{-s}-e^{-1}} dB(s)\right) \end{aligned} \tag{12}$$
has Gaussian distribution with expectation vector $(0, 0)$ and covariance matrix
$$C := \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ . \tag{13}$$
Therefore,
$$\begin{aligned} f_Y(x, y) &= \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left\{-\frac{\left\langle C^{-1}(x, y), (x, y)\right\rangle}{2}\right\} \\ &= \frac{1}{\sqrt{(2\pi)^2 (ac - b^2)}} \exp\left\{-\frac{1}{2(ac - b^2)}\left(bx^2 - 2bxy + cy^2\right)\right\} \ . \end{aligned} \tag{14}$$

■

Exercise 2 *Let* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\right)$ *be the filtered Wiener space. Consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t, X_0) = X_0 + t + 2\int_0^t \sqrt{X(s)} dB(s) \ ,$$
$$\begin{cases} dX(t) = dt + 2\sqrt{X(t)} dB(t) \\ X(0) = X_0 \end{cases} \ .$$

where $(B(t), t \geq 0)$ is the Brownian motion and X_0 is a r.v. independent of the filtration generated by the Brownin motion.

What is the probability distribution of the solution at a given time t?

Solution: Set $X(t, X_0) = f(t, B(t))$ where $f \in C^{1,2}(\mathbb{R}^2)$. By the Itô's Lemma we get

$$df(B(t)) = \left[\left(\frac{\partial}{\partial t} f\right)(t, B(t)) + \frac{1}{2}\left(\frac{\partial^2}{\partial y^2} f\right)(t, B(t))\right] dt + \left(\frac{\partial}{\partial y} f\right)(t, B(t))\, dB(t) \ .$$

Hence,

$$X_0 + t + 2\int_0^t \sqrt{X(s)} dB(s) = X_0 + \int_0^t ds + 2\int_0^t \sqrt{f(B(s), X_0)} dB(s)$$
$$= f(0,0) + \int_0^t ds \left[\left(\frac{\partial}{\partial t} f\right) + \frac{1}{2}\left(\frac{\partial^2}{\partial y^2} f\right)\right](s, B(s)) + \int_0^t \left(\frac{\partial}{\partial y} f\right)(s, B(s))\, dB(s) \ ,$$

which implies

$$\begin{cases} \left(\frac{\partial}{\partial y} f\right)(t, y) = 2\sqrt{f(t, y)} \\ \left(\frac{\partial}{\partial t} f\right)(t, y) + \frac{1}{2}\left(\frac{\partial^2}{\partial y^2} f\right)(t, y) = 1 \\ f(0,0) = X_0 \end{cases} .$$

Differentiating the first equation with respect to y we get

$$\left(\frac{\partial^2}{\partial y^2} f\right) = \frac{1}{\sqrt{f(t, y)}}\left(\frac{\partial}{\partial y} f\right)(t, y) = 2$$

which, inserted in the second equation gives $\left(\frac{\partial}{\partial t} f\right)(t, y) = 0$. Hence, $f(t, y) = y^2 + X_0$, that is $X(t, X_0) = B^2(t) + X_0$. ■