Solution of the execises of the exam of the course *Probability and Stochastic Processes*

a.y. 2022/2023
11/30/2023

Exercise 1 *Let* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ *be the filtered Wiener space. We consider the stochastic process described by the Itô Stochastic Differential Equation*

$$X(t,X_0)=X_0+\int_0^t ds\left[\sqrt{s}X(s)+e^{-s}\right]+\int_0^t s dB(s) \ , \quad (1)$$
$$dX(t)=\left[\sqrt{s}X(t)+e^{-t}\right]dt+t dB(t) \ .$$

where $\{B(t)\}_{t\geq 0}$ *is the Brownian motion.*

1. *Solve the equation (1) assuming the initial datum* $X_0=0$.
2. *Compute the probability distribution of the stochastic process* $(X(t,X_0)\ ,\ t\geq 0)$.
3. *Compute the density of the random vector* $(X(2,X_0),X(1,X_0))$.

Solution:

1. See Exercise 1 of the exam of july the 7th 2018. The solution of equation (1) is
$$X(t,X_0)=e^{\frac{2}{3}t^{\frac{3}{2}}}\left[X_0+\int_0^t dse^{-\left(\frac{2}{3}s^{\frac{3}{2}}+s\right)}+\int_0^t se^{-\frac{2}{3}s^{\frac{3}{2}}}dB(s)\right] \ .$$
2. Since $X_0=0$, from the previous equation we have that setting $\forall t>0, X(t,0)=:X(t)$,
$$\begin{aligned}\mu(t) \ &:= \mathbb{E}[X(t)]=e^{\frac{2}{3}t^{\frac{3}{2}}}\int_0^t dse^{-\left(\frac{2}{3}s^{\frac{3}{2}}+s\right)}\\ \sigma^2(t) \ &:= \mathbb{E}\left[(X(t)-\mathbb{E}[X(t)])^2\right]=\mathbb{E}\left[e^{\frac{4}{3}t^{\frac{3}{2}}}\left(\int_0^t se^{-\frac{2}{3}s^{\frac{3}{2}}}dB(s)\right)^2\right]\\ &= e^{\frac{4}{3}t^{\frac{3}{2}}}\int_0^t dss^2e^{-\frac{4}{3}s^{\frac{3}{2}}} \ .\end{aligned}$$
Therefore, $\forall t>0, X(t,X_0)\stackrel{d}{=}N\left(\mu(t),\sigma^2(t)\right)$.

3. Since

$$\begin{aligned} Cov\left[X\left(t,X_0\right),X\left(s,X_0\right)\right] &= \mathbb{E}\left[\left(X\left(t,X_0\right)-\mathbb{E}\left[Y\left(t,X_0\right)\right]\right)\left(Y\left(s,X_0\right)-\mathbb{E}\left[Y\left(s,X_0\right)\right]\right)\right] \\ &= e^{\frac{2}{3}\left(t^{\frac{3}{2}}+s^{\frac{3}{2}}\right)}\mathbb{E}\left[\int_0^{t\wedge s} d\tau\tau^2 e^{-\frac{4}{3}\tau^{\frac{3}{2}}}\right] = c\left(t,s\right) \end{aligned}$$

The random vector

$$X := \left(X\left(2,X_0\right),X\left(1,X_0\right)\right)$$

has Gaussian distribution with expectation vector

$$\mu = \left(\mu\left(2\right),\mu\left(1\right)\right) = \left(\mu_1,\mu_2\right)$$

and covariance matrix

$$C := \begin{pmatrix} \sigma^2\left(2\right) & c\left(1,2\right) \\ c\left(1,2\right) & \sigma^2\left(1\right) \end{pmatrix} = \begin{pmatrix} a & cb \\ cb & b \end{pmatrix} .$$

where $c := e^{\frac{2}{3}\left(2^{\frac{3}{2}}+1\right)}$. Therefore,

$$f_Y\left(x,y\right) = \frac{1}{\sqrt{\left(2\pi\right)^2 \det C}} \exp\left\{-\frac{\left\langle C^{-1}\left(x-\mu_1,y-\mu_2\right),\left(x-\mu_1,y-\mu_2\right)\right\rangle}{2}\right\}$$

where $\det C = b\left(a-c^2b\right)$ and

$$C^{-1} = \frac{1}{b\left(a-c^2b\right)}\begin{pmatrix} b & -cb \\ -cb & a \end{pmatrix} .$$

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Exercise 2 *Let* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ *be the filtered Wiener space. Consider the stochastic process* $\left(X\left(t,X_0\right)\ ,\ t\geq 0\right)$ *such that* $\forall t\geq 0, X\left(t,X_0\right) := \left(B\left(t\right)+X_0^{\frac{1}{3}}\right)^3$.

1. *Write the Itô Stochastic Differential Equation satisfied by* $\left(X\left(t,X_0\right)\ ,\ t\geq 0\right)$.
2. *Let* $X_0 = 0$. *Compute, the expectation and the variance of* $\left(X\left(t\right)\ ,\ t\geq 0\right)$, *where* $\forall t\geq 0, X\left(t\right) := X\left(t,X_0\right)$.

Solution:

1. Let $c := X_0^{\frac{1}{3}} \in \mathbb{R}$ and $\mathbb{R}\ni x\mapsto g\left(x\right) := \left(x+c\right)^3 \in \mathbb{R}$. Then, $X\left(t,X_0\right) = f\left(t,B\left(t\right)\right) = g\left(B\left(t\right)\right)$. Since

$$\begin{aligned} \partial_t f\left(t,x\right) &= 0\ , \\ \partial_x f\left(t,x\right) &= 3\left(x+c\right)^2\ , \\ \partial_x^2 f\left(t,x\right) &= 6\left(x+c\right)\ , \end{aligned}$$

because $B\left(t\right)+X_0^{\frac{1}{3}} = X^{\frac{1}{3}}\left(t,X_0\right)$, applying the Itô Lemma, we get

$$X\left(t,X_0\right) = X_0^{\frac{1}{3}} + 3\int_0^t ds X^{\frac{1}{3}}\left(s\right) + 3\int_0^t X^{\frac{2}{3}}\left(s\right)dB\left(s\right) .$$

2. By definition $X(t) = B^3(t)$. Hence, since $B(t) \overset{d}{=} \sqrt{t}B(1)$, where $B(1) \overset{d}{=} N(0,1)$, because the density of the standard normal r.v. is an even function,

$$\mathbb{E}[X(t)] = \mathbb{E}\left[B^3(t)\right] = t^{\frac{3}{2}}\mathbb{E}\left[B^3(1)\right] = t^{\frac{3}{2}} \int_{\mathbb{R}} dx x^3 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} = 0 \ .$$

Therefore, the variance of $X(t)$ is equal to

$$\mathbb{E}\left[X^2(t)\right] = \mathbb{E}\left[B^6(t)\right] = t^3\mathbb{E}\left[B^6(1)\right] = t^3 \int_{\mathbb{R}} dx x^6 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} = 15t^3 \ .$$

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