Solution of the execises of the exam of the course *Probability and Stochastic Processes*

a.y.2024/2025
01/31/2025

Exercise 1 *Let* $\{U_i\}_{i\geq 1}$ *a sequence of i.i.d.r.v.'s on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $U_1 \stackrel{d}{=} Unif\,[0,1]$. *Let* $\mathcal{F}_0$ *be the trivial* σ*algebra and, for any* $n \geq 1$, *let* $\mathcal{F}_n$ *be the* σ*algebra generated by the random vector* $(U_1, .., U_n)$. *Given* $\theta \in (0,1)$, *set* $X_0 := p \in (0,1)$ *and* $\forall n \geq 1, X_{n+1} := \theta X_n + (1-\theta)\,\mathbf{1}_{[0,X_n]}(U_{n+1})$.

1. *Prove* $\forall n \geq 0, X_n \in [0,1]$.
2. *Prove that* $\{X_n\}_{n\geq 0}$ *is an* L^2*-martingale w.r.t.* $\{\mathcal{F}_n\}_{n\geq 0}$.
3. *Compute* $\mathbb{E}\left[(X_{n+1}-X_n)^2\right]$, *compute its limit as* $n \uparrow \infty$ *and deduce the distribution of the limit* X_∞ *of the terminal value of* $\{X_n\}_{n\geq 0}$.

Solution:

1. Since by definition $X_0 \in [0,1]$, then

$$0 < \theta p \leq X_1 = \theta p + (1-\theta)\,\mathbf{1}_{[0,p]}(U_{n+1}) \leq \theta p + (1-\theta) < 1\ .$$

   Hence, assuming $X_n \in [0,1]$, we get

$$0 \leq X_{n+1} \leq \theta X_n + (1-\theta) \leq 1$$

   and the statement follows by induction.

2. By the previous statement $\forall n \geq 0, X_n \in L^1$, therefore we are left with the proof of the martingale property. But

$$\begin{aligned} \mathbb{E}[X_{n+1}|\mathcal{F}_n] \quad &: \quad = \theta X_n + (1-\theta)\,\mathbb{E}\left[\mathbf{1}_{[0,X_n]}(U_{n+1})\,|\mathcal{F}_n\right] \\ &= \quad \theta X_n + (1-\theta)\,X_n = X_n\ . \end{aligned}$$

   Moreover $\forall n \geq 0, \mathbb{E}\left[X_n^2\right] \leq 1$ hence Prove that $\{X_n\}_{n\geq 0}$ is an L^2-martingale.

3.
$$\mathbb{E}\left[\left(X_{n+1}-X_n\right)^2\right]=\left(1-\theta\right)^2\mathbb{E}\left[\left(X_n-\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right)^2\right]\ .$$

But

$$\mathbb{E}\left[\left(X_n-\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right)^2\right]=\mathbb{E}\left[X_n^2\right]-2\mathbb{E}\left[X_n\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right]+\mathbb{E}\left[\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right]$$

and

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)|\mathcal{F}_n\right]\right]=\mathbb{E}\left[X_n\right]\ ,\\ \mathbb{E}\left[X_n\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[X_n\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)|\mathcal{F}_n\right]\right]=\\ &= \mathbb{E}\left[X_n\mathbb{E}\left[\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)|\mathcal{F}_n\right]\right]=\mathbb{E}\left[X_n^2\right]\ ,\end{aligned}$$

which implies

$$\mathbb{E}\left[\left(X_n-\mathbf{1}_{[0,X_n]}\left(U_{n+1}\right)\right)^2\right]=\mathbb{E}\left[X_n\right]-\mathbb{E}\left[X_n^2\right]=\mathbb{E}\left[X_n\left(1-X_n\right)\right]$$

so that
$$\mathbb{E}\left[\left(X_{n+1}-X_n\right)^2\right]=\left(1-\theta\right)^2\mathbb{E}\left[X_n\left(1-X_n\right)\right]\ .$$

Since $\{X_n\}_{n\geq0}$ is an L^2-martingale, it is convergent in L^2 to a $\mathcal{F}_\infty$-measurable r.v. X_∞ such that $X_n=\mathbb{E}\left[X_\infty|\mathcal{F}_n\right]$. Thus

$$\begin{aligned}0 &\leq \left(1-\theta\right)^2\lim_{n\uparrow\infty}\mathbb{E}\left[X_n\left(1-X_n\right)\right]=\lim_{n\uparrow\infty}\mathbb{E}\left[\left(X_{n+1}-X_n\right)^2\right]\\ &\leq \lim_{n\uparrow\infty}\mathbb{E}\left[\left(\left(X_{n+1}-X_\infty\right)+\left(X_n-X_\infty\right)\right)^2\right]\\ &\leq \lim_{n\uparrow\infty}\left(\left\|X_{n+1}-X_\infty\right\|_{L^2}+\left\|X_n-X_\infty\right\|_{L^2}\right)^2=0\ ,\end{aligned}$$

which, since X_∞ is non negative, implies that $X_\infty\left(1-X_\infty\right)=0,\mathbb{P}$-a.s.. Furthermore, since L^2-martingales are regular $\{X_n\}_{n\geq0}$ is convergent to $X_\infty,\mathbb{P}$-a.s. and because $\mathbb{E}\left[X_n\right]=\mathbb{E}\left[X_0\right]=p>0$,we get that $X_\infty\overset{d}{=}Ber\left(p\right)$.

■

Exercise 2 *Let* $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P}\right)$ *be the filtered Wiener space. Solve the stochastic process described by the Itô Stochastic Differential Equation*

$$\begin{aligned}X\left(t,X_0\right)&=X_0+\int_0^t ds\frac{s}{2}X\left(s\right)+\int_0^t 2\sqrt{s}X\left(s\right)dB\left(s\right)\ ,\\ dX\left(t\right)&=\frac{t}{2}X\left(t\right)dt+2\sqrt{t}X\left(t\right)dB\left(t\right)\ .\end{aligned}\tag{1}$$

where $\left(B\left(t\right),t\geq0\right)$ *is the Brownian motion.*

1. *Compute the variance of the stochastic process* $\left(X\left(t,1\right)\ ,\ t\geq0\right)$.

Solution: The equation (1) sia a Itô's SDE with multiplicative noise.

1. Setting
$$Y(t) = f(t, X(t)) := \log \frac{X(t)}{X_0} \tag{2}$$
and computing the Itô's differential of $Y(t)$, since $f(t, x) = \log \frac{x}{X_0}$ and
$$\partial_t f(t, x) = 0 \ , \tag{3}$$
$$\partial_x f(t, x) = \frac{X_0}{x} \ , \tag{4}$$
$$\partial_x^2 f(t, x) = -\frac{X_0}{x^2} \ , \tag{5}$$
we get
$$dY(t) = \left(\frac{1}{2}t - 2t\right) dt + 2\sqrt{t} dB(t) \ , \ Y(t) = -\frac{3}{4}t^2 + \int_0^t 2\sqrt{s} dB(s) \ . \tag{6}$$
That is, taking into account that $Y(0, X(0)) = 0$,
$$X(t, X_0) = X_0 e^{-\frac{3}{4}t^2 + \int_0^t 2\sqrt{s} dB(s)} \ . \tag{7}$$

2. Setting $X_0 = 1$, from (1) we get
$$\mathbb{E}[X(t, 1)] = 1 + \mathbb{E}\left[\int_0^t ds \frac{s}{2} X(s)\right] + \mathbb{E}\left[\int_0^t 2\sqrt{s} X(s) dB(s)\right] \ , \tag{8}$$
that is, $\mu_X(t) := \mathbb{E}[X(t, 1)]$, satisfies the Chauchy's problem
$$\begin{cases} \frac{d}{dt}\mu_X(t) = \frac{t}{2}\mu_X(t) \\ \mu_X(0) = 1 \end{cases} \ . \tag{9}$$
Hence,
$$\mu_X(t) = e^{\int_0^t ds \frac{s}{2}} = e^{\frac{t^2}{4}} \ . \tag{10}$$
Moreover, putting $Y(t) := f(t, X(t)) = X^2(t)$ and computing its Itô's differential we have
$$Y(t) = 1 + 5\int_0^t ds s X^2(s) + \int_0^t 4X^2(s)\sqrt{s} dB(s) \ . \tag{11}$$
Thus, denoting by $q_X(t) := \mathbb{E}[Y(t)] = \mathbb{E}\left[X^2(t)\right]$, taking the expectation of both sides of the preceding Itô's equations and taking the derivative w.r.t. t we obtain that q_X is the solution of the Cauchy's problem
$$\begin{cases} \frac{d}{dt}q_X(t) = 5t q_X(t) \\ q_X(0) = 1 \end{cases} \tag{12}$$
i.e.
$$q_X(t) = e^{\frac{5}{2}t^2} \ . \tag{13}$$
Therefore, the variance of $X(t, 1)$ is
$$\mathbb{E}\left[X^2(t, 1)\right] - \mathbb{E}^2[X(t, 1)] = q_X(t) - \mu_X^2(t) = e^{\frac{5}{2}t^2} - e^{\frac{t^2}{2}} \ . \tag{14}$$

■

Exercise 3 *Compute the characteristic function of the random vector* $(\log Y(1,1), \log Y(2,1))$, *where* $(Y(t,1)\ ,\ t \geq 0)$ *is the stochastic process solution of the homogeneuous Itô SDE associated to the equation (1).*

Solution: The distribution of the random vector

$$\begin{aligned} Y \quad &: \quad = (\log X(1,1), \log X(2,1)) \\ &= \left(-\frac{3}{4} + 2\int_0^1 \sqrt{t} dB(t), -3 + 2\int_0^2 \sqrt{t} dB(t)\right) \end{aligned} \tag{15}$$

is Gaussian with parameters

$$\mu = \left(-\frac{3}{4}, -3\right) \tag{16}$$

and, since

$$a(t) := 4\int_0^t dss = 2t^2\ , \tag{17}$$

covariance matrix

$$C := \begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix}\ . \tag{18}$$

Thus, $\forall \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$,

$$\begin{aligned} \varphi_Y(\lambda) \quad &: \quad = \mathbb{E}\left[e^{i\langle \lambda, Y\rangle}\right] = e^{i\langle \mu, \lambda\rangle - \frac{1}{2}\langle C\lambda, \lambda\rangle} \\ &= e^{-i\left(\frac{3}{4}\lambda_1 + 3\lambda_2\right)} \exp\left[-\frac{1}{2}\left(2\lambda_1^2 + 4\lambda_1\lambda_2 + 8\lambda_2^2\right)\right] \\ &= e^{-i\left(\frac{3}{4}\lambda_1 + 3\lambda_2\right)} \exp\left[-\left(\lambda_1^2 + 2\lambda_1\lambda_2 + 4\lambda_2^2\right)\right]\ . \end{aligned} \tag{19}$$

■