Processing of Declarative Knowledge –Complexity Issues–

Francesco Ricca

Computational Intelligence Curriculum
Institute of Information Systems

ASP Basics

ASP:

Datalog $\leftarrow$ done!

+ Default negation $\leftarrow$ done!
+ Disjunction $\leftarrow$ done!
+ Integrity Constraints $\leftarrow$ done!
+ Weak Constraints $\leftarrow$ done!
+ Aggregate atoms $\leftarrow$ done!

Complexity of Reasoning with ASP

- Complete the picture of what can be done...
- the computational cost

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Complexity Issues

Understand ASP Complexity

- know language subclasses
- understand the solving process (later)

Consider Data-Complexity

- programs offer uniform solutions over instances
- i.e., fixed encoding + facts
- *basic language:* no aggregates, no functions

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Decision Problems

Main Decision Problems:

1. Answer Set Checking
   - *Given an ASP program P and an interpretation I, is I an answer set of P?*
2. Brave Reasoning
   - *Given an ASP program P and a ground atom a, is a true in some answer set of P?*
3. Cautious Reasoning
   - *Given an ASP program P and a ground atom a, is a true in all answer sets of P?*

Syntactic Restrictions

Stratified Programs (i.e., no recursion trough negation)

P is (locally) stratified if there is a level mapping $\|.\|_s$ of P such that for every rule r of P:

1. For any l in the positive body of r, and for any l' in the head of r, $\|l\|_s \leq \|l'\|_s$
2. For any not l in the negative body, and for any l' in the head of r, $\|l\|_s < \|l'\|_s$

Level Mapping: a function $\|.\|$ from ground literals to positive integers.

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Example (Stratified Program)

$$p(a) \mid p(c) \text{ :- } \mathrm{not}\ q(a).$$
$$p(b) \text{ :- } \mathrm{not}\ q(b).$$

is stratified since:

$$\|p(a)\|_s = 2,\ \|p(b)\|_s = 2,\ \|p(c)\|_s = 2$$
$$\|q(a)\|_s = 1,\ \|q(b)\|_s = 1$$

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Example (Unstratified Program)

$$p(a) \mid p(c) :- \mathrm{not}\ q(b).$$
$$q(b) :- \mathrm{not}\ p(a)$$

No stratified level mapping exists!

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Stratification Theorems:

- Stratified non-disjunctive programs admit at most one answer set
- The answer set of a stratified and non-disjunctive program is computable in Polynomial Time.

Syntactic Restrictions (2)

Head-Cycle Free (HCF) Programs

P is head-cycle free if there is a level mapping $\|.\|_h$ of P such that for every rule r of P:

1. For any l in the positive body of r, and for any l' in the head of r, $\|l\|_h \leq \|l'\|_h$
2. For any pair l, l' of atoms in the head of r, $\|l\|_h <> \|l'\|_h$

Example (HCF Program)

$a \mid b.$
$a \text{ :- } b.$

is HCF since:

$$\|a\|_h = 2; \quad \|b\|_h = 1$$

Syntactic Restrictions (2)

Head-Cycle Free (HCF) Programs

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Example (Non-HCF Program)

$$a \mid b.$$
$$a :\!-\, b.$$
$$b :\!-\, a.$$

Recursion trough disjunction $\rightarrow$ Non HCF level mapping exists!

Syntactic Restrictions (2)

Head-Cycle Free (HCF) Programs

P is head-cycle free if there is a level mapping $\|.\|_h$ of P such that for every rule r of P:

1. For any l in the positive body of r, and for any l' in the head of r, $\|l\|_h \leq \|l'\|_h$
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Head-Cycle Free Theorem:

- Every head-cycle free program P is equivalent to an non-disjunctive program where disjunction is "shifted" to the body
- e.g., $a \mid b :- c.$ is equivalent to: $b :- \text{not } a, c.$ $\quad a :- \text{not } b, c.$

Intuitive explanation

Three main sources of complexity:

- the exponential number of answer set “candidates”
- the difficulty of checking whether a candidate M is an answer set of P (the minimality of M can be disproved by exponentially many subsets of M)
- the difficulty of determining the optimality of the answer set w.r.t. the violation of the weak constraints

Complexity of Answer Set Checking

	{}	not_s	not
{}	P	P	P
V_h	P	P	P
V	coNP	coNP	coNP

Complexity of Brave Reasoning

	{}	not_s	w	w,not_s	not	w, not
{}	P	P	P	P	NP	Δ^P_2
V_h	NP	NP	Δ^P_2	Δ^P_2	NP	Δ^P_2
V	Σ^P_2	Σ^P_2	Δ^P_3	Δ^P_3	Σ^P_2	Δ^P_3

Completeness under Logspace reductions

Complexity of Cautious Reasoning

	{}	not_s	w	w,not_s	not	w, not
{}	P	P	P	P	coNP	Δ^P_2
V_h	coNP	coNP	Δ^P_2	Δ^P_2	coNP	Δ^P_2
V	coNP	Π^P_2	Δ^P_3	Δ^P_3	Π^P_2	Δ^P_3

Note that < V, {} > is “only” coNP-complete!

Exercise 2QBF

Given a propositional formula $\exists x \forall y \Phi(x, y)$ in DNF, compute an assignment to y-variables that satisfies Φ for all assignments to x-variables, if it exists.

Write a disjunctive logic program $P(\exists x \forall y \Phi(x, y))$ such that answer sets of $P(\exists x \forall y \Phi(x, y))$ correspond to satisfying assignments of $\exists x \forall y \Phi(x, y)$

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