Univerität Basel

Herbsemester 2012

Master course A. Surroca - L. Paladino

Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.12

12.1. We consider the double series

$$G_2(z) = \sum_n \sum_m{}' \frac{1}{(m+nz)^2};$$
$$G(z) = \sum_m \sum_n{}' \frac{1}{(m+nz)^2};$$
$$H_2(z) = \sum_n \sum_m{}' \frac{1}{(m-1+nz)(m+nz)};$$
$$H(z) = \sum_m \sum_n{}' \frac{1}{(m-1+nz)(m+nz)};$$

where the symbol $\sum'$ indicates that (m, n) runs through all $m \in \mathbb{Z}, n \in \mathbb{Z}$, with $(m, n) \neq (0, 0)$ for G_2 and G and $(m, n) \notin \{(0, 0), (1, 0)\}$, for H_2 and H. (Notice the order of the summations!)

a) Prove that both H_2 and H converge and that $H_2 = 2$, $H = 2 - (2\pi i)/z$.

Hint: use the formulas

$$\frac{1}{(m-1+nz)(m+nz)} = \frac{1}{m-1+nz} - \frac{1}{m+nz}$$

$$\pi\cot\pi\tau = \frac{1}{\tau} + \sum_{d=1}^{\infty}(\frac{1}{\tau-d} + \frac{1}{\tau+d});$$

$$\pi\cot\pi\tau = \pi i - 2\pi i \sum_{m=0}^{\infty} e^{2m\tau\pi i}, \quad \text{with } m \in \mathbb{N}.$$

b) Prove that $G_2 - H_2 = G - H$.

Hint: use the formula

$$\frac{1}{(m-1+nz)(m+nz)} - \frac{1}{(m+nz)^2} = \frac{1}{(m+nz)^2(m-1+nz)}$$

c) Prove that $G_1(-1/z) = z^2 G(z)$.

d) Prove that

$$G_1(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

where, as usual, $q = e^{2\pi i z}$.

Hint: use a similar proof to the one needed to show the Fourier expansion of the Eisenstein series:

$$G_k(z) = 2\zeta(z) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \text{for } k \geq 4.$$

e) Define $F(z) = q \prod_{n=1}^{\infty}(1-q^n)^{24}$. Prove that

$$\frac{F'}{F}(z) = 2\pi i(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n).$$