Univerität Basel

Herbsemester 2012

Master course A. Surroca - L. Paladino

Some topics on modular functions, elliptic functions and transcendence theory

Sheet of exercises n.8

8.1. Let

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Prove that this series converges uniformly on compact sets.

8.2. a) Prove that

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \sum_{m=1}^{\infty} \frac{m+1}{\omega^{m+2}} z^m,$$

for all $z \in \mathbb{C} \setminus \Lambda$ and $\omega \in \Lambda \setminus \{0\}$.

b) Prove the Laurent expansion of $\mathcal{P}$:

$$\mathcal{P}_\Lambda(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2}(\Lambda) z^{2n}, \qquad \text{for all } z \in \mathbb{C}.$$

8.3. Prove that $\mathcal{P}$ satisfies the differential equation

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - g_2 \mathcal{P}(z) - g_3.$$

8.4. Express $\mathcal{P}'''(z)$ in terms of $\mathcal{P}(z)$ and $\mathcal{P}'(z)$.

8.5. Prove that

$$\mathcal{P}''(z) = 6\mathcal{P}(z)^2 - \frac{1}{2}g_2,$$

by repeating the argument used in Exercise **8.3**.

8.6 Let $\Lambda = \mathbb{Z}_{\omega_1} \oplus \mathbb{Z}_{\omega_2}$ and $\omega_3 = \omega_1 + \omega_2$. Prove that the zeroes of $\mathcal{P}'$ are the points $\omega_k/2$ modulo Λ and they are simple.