

Univerität Basel
Herbsemester 2012
Master course A. Surroca - L. Paladino

*Some topics on modular functions, elliptic functions
and transcendence theory*

Sheet of exercises n.9

For all the sheet, let $\Lambda = \mathbb{Z}_{\omega_1} \oplus \mathbb{Z}_{\omega_2}$ be a complex lattice, let $\wp = \wp_{\Lambda}$ be the Weierstrass \wp -function associate to Λ . Set $\omega_3 = \omega_1 + \omega_2$ and $e_k := \wp(\omega_k/2)$, for $i \in \{1, 2, 3\}$.

9.1. Prove that $e_1 + e_2 + e_3 = 0$.

9.2. Prove the two duplication formulas satisfied by the Weierstrass \wp -function, for $u \in \mathbb{C}/\Lambda$, $2u \notin \mathbb{C}/\Lambda$:

a)

$$\wp(2u) = \frac{1}{4} \left(\frac{\wp''(u)}{\wp'(u)} \right)^2 - 2\wp(u);$$

b)

$$\wp(2u) = \frac{(3\wp(u)^2 - \frac{1}{4}g_2)^2}{4\wp(u)^3 - g_2\wp(u) - g_3} - 2\wp(u).$$

9.3. Prove that

$$\wp\left(z + \frac{1}{2}\omega_1\right) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(z) - e_1}$$

9.4. Prove that

$$\wp(u+v) - \wp(u-v) = -\frac{\wp'(u)\wp'(v)}{(\wp(u) - \wp(v))^2}.$$

- 9.5. a)** Prove that, if n is an integer, then $\wp(nu)$ can be expressed as a rational function of $\wp(u)$.
- b)** Show, in particular, that

$$\wp(3u) = \wp(u) + \frac{\wp'(u)^2(\wp'(u)^4 - \Psi(u)\wp''(u))}{\Psi(u)^2},$$

where $\Psi(u) = \wp'(u)^2(\wp(u) - \wp(2u))$.