Efficient Problem Solving on Tree Decompositions using Binary Decision Diagrams

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Motivation

Solve your favorite intractable (graph) problem...

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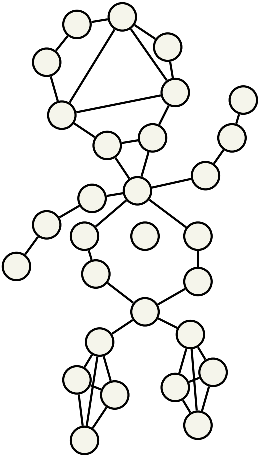

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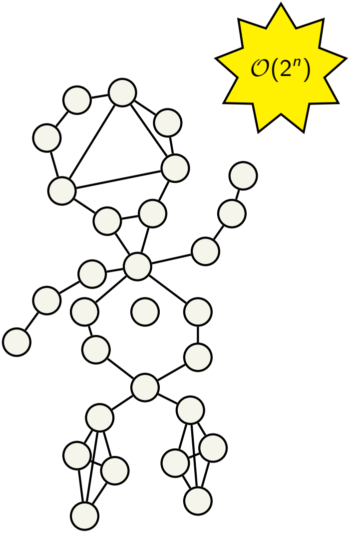

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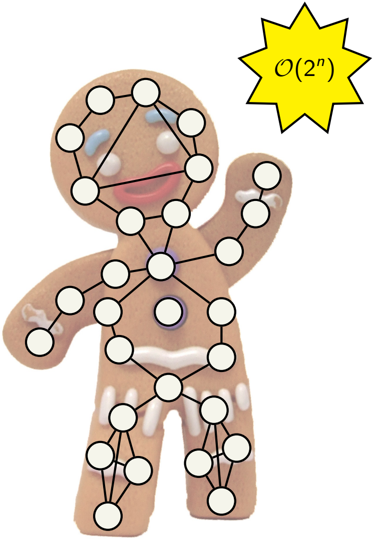

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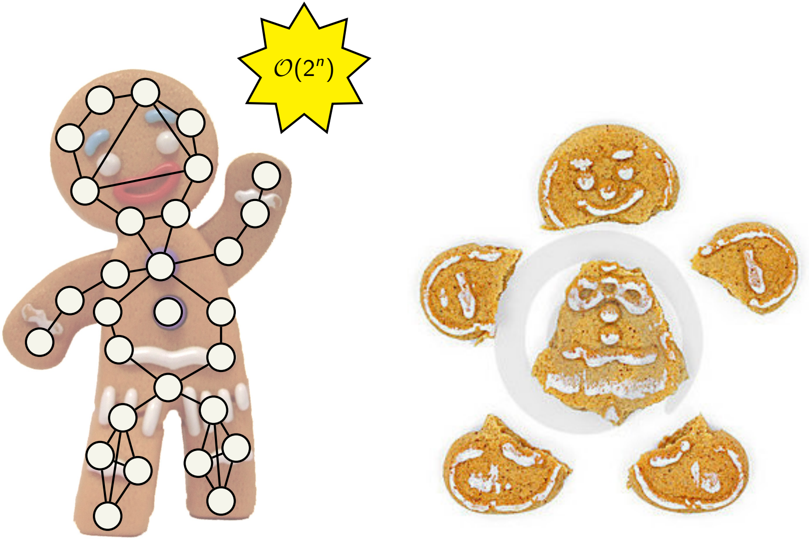

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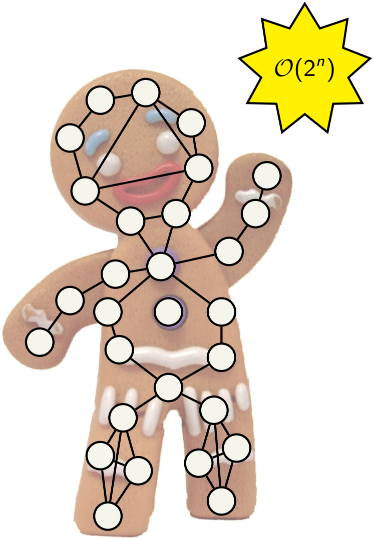

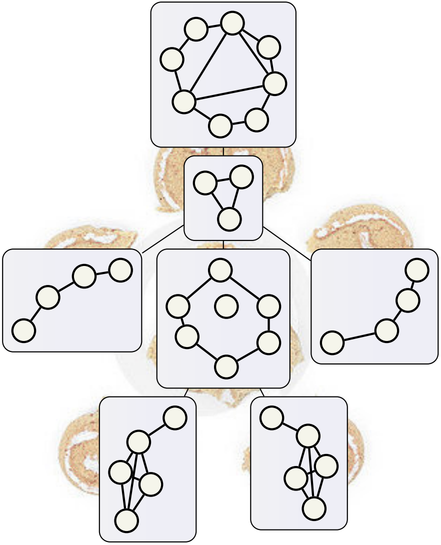

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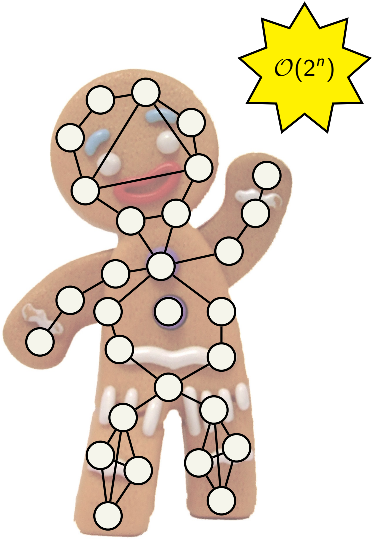

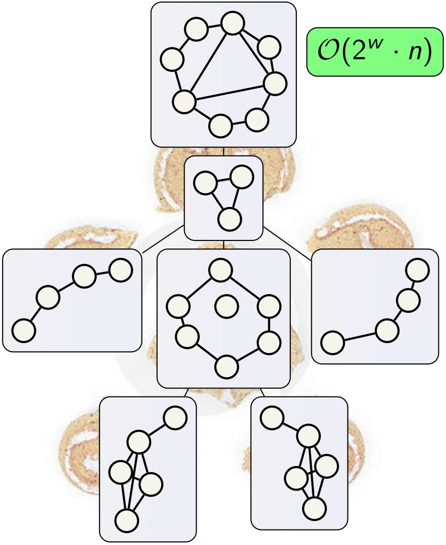

Motivation

Dynamic programming on tree decompositions in a nutshell

Basic idea

- For hard problems exploit structural properties of instance
- Confine complexity to a parameter
- Many problems are fixed-parameter tractable (fpt) w.r.t. tree-width w, i.e. solvable in time

$$f(w) \cdot n^{\mathcal{O}(1)}$$

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General Approach

1. Decompose instance
2. Solve partial problems
3. Get result at final node

Motivation

Practical realization

- Intermediate results stored in *tables*
- Computation via *manipulation of rows*

Problem: Large memory footprint

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Our paradigm

- *Native* support for efficient storage
- *Logic-based* algorithm specifications
- Algorithms define how *sets* of partial solutions are computed

We use **Binary Decision Diagrams** as data structure

Background

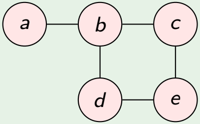
Tree decompositions

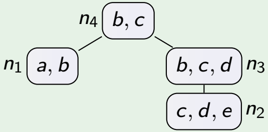
Definition

A *tree decomposition* is a tree obtained from an arbitrary graph s.t.

1. Each vertex must occur in some *bag*.
2. For each edge, there is a bag containing both endpoints.
3. If vertex v appears in bags of nodes n_0 and n_1, then v is also in the bag of each node on the path between n_0 and n_1.

Example





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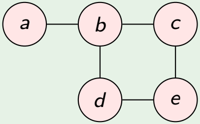

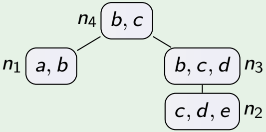

- *Width*: Size of largest bag minus 1
- *Tree-width*: Minimum width over all possible tree decompositions

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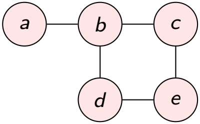
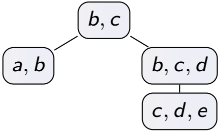
Tree decompositions

Definition

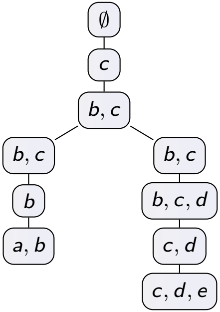
Each node in a *normalized tree decomposition* is of one of the following types: leaf, introduction, removal, or join node.



Input instance



Tree decomposition



Normalized tree decomposition

Background

Binary Decision Diagrams (BDDs)

- Data structure for storing Boolean functions
- Representation as rooted DAG
- Reduced Ordered BDDs [Bryant, 1986] particularly space efficient

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Let formula $\phi = (a \land b \land c) \lor (a \land \neg b \land c) \lor (\neg a \land b \land c)$.

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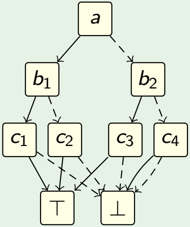

OBBD of ϕ.

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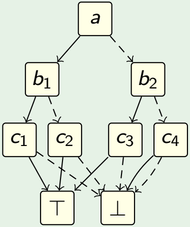

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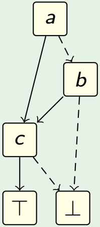

ROBBD of ϕ.

Background

Binary Decision Diagrams (BDDs)

Advantages of BDDs

- Well-studied concept (applied to model checking, planning, software verification, . . .)
- Efficient implementations available
- Memory-efficient storage handled directly by data structure
- Logic-based algorithm specification

Background

Binary Decision Diagrams (BDDs)

BDDs support

- Standard logical operators ($\wedge$, $\vee$, $\neg$, $\leftrightarrow$, $\ldots$)
- *Existential quantification* ($\exists V \mathcal{B}$)
- *Restriction* and *renaming* ($\mathcal{B}[v/\cdot]$ where $\cdot \in \{\top, \bot, v'\}$)

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Size of ROBDDs

- Bounded by $\mathcal{O}(2^{|V_{\mathcal{B}}|})$
- Heavily depends on variable ordering
- Finding optimal ordering is NP-complete [Bollig and Wegener, 1996]
- But there are good heuristics (e.g., [Rudell, 1993])

In practice often only polynomially large! [Friedman and Supowit, 1987]

Dynamic Programming using BDDs

Concept comparison

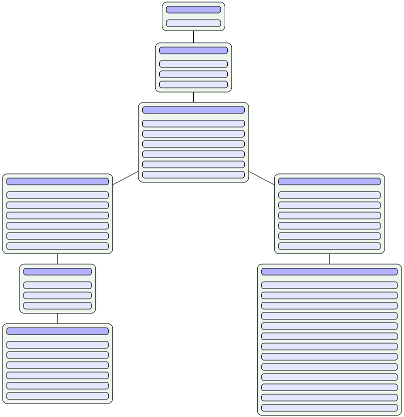

Table-based Dynamic Programming

Dynamic Programming using BDDs

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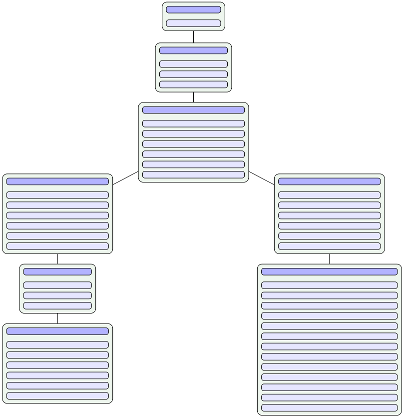

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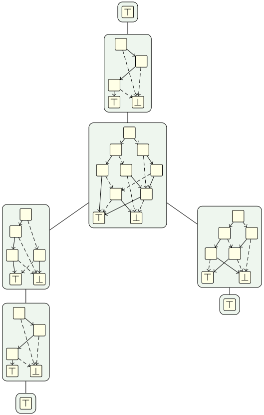


BDD-based Dynamic Programming

Dynamic Programming using BDDs

Approach

Preparation

- Specify problem-dependent BDD manipulation operations $\mathcal{B}^*$
- Distinguish between node types, here: $* \in \{l, i, r, j\}$ (leaf, introduction, removal, join)

Dynamic Programming using BDDs

Approach

Preparation

- Specify problem-dependent BDD manipulation operations $\mathcal{B}^*$
- Distinguish between node types, here: $* \in \{l, i, r, j\}$ (leaf, introduction, removal, join)

Solve problem

1. Decompose instance to obtain tree decomposition $\mathcal{T}$
2. Traverse $\mathcal{T}$ in post-order and for each node n in $\mathcal{T}$, compute $\mathcal{B}^*_n$ based on node type $*$
3. In root node r of $\mathcal{T}$, either $\mathcal{B}_r = \top$ or $\mathcal{B}_r = \bot$ holds

3-Colorability

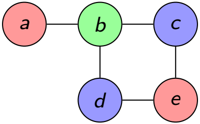
Problem

Given a graph $G = (V, E)$, is G 3-colorable, i.e.:

- each vertex gets assigned exactly one color, and
- neighboring vertices have different colors?

Variables

Color assignment: c_x for all $c \in C = \{r, g, b\}, x \in V$



3-Colorability

$$\mathcal{B}_n^l = \bigwedge_{c \in C} \bigwedge_{\{x,y\} \in E_n} \neg(c_x \wedge c_y) \wedge \bigwedge_{x \in X_n} (r_x \vee g_x \vee b_x) \wedge$$

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$$\mathcal{B}^i_n = \mathcal{B}_{n'} \wedge \bigwedge_{c \in C} \bigwedge_{\{x,u\} \in E_n} \neg(c_x \wedge c_u) \wedge (r_u \vee g_u \vee b_u) \wedge$$
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(Here, u is the introduced vertex.)

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$$\mathcal{B}_n^r = \exists r_u g_u b_u [\mathcal{B}_{n'}]$$

(Here, u is the removed vertex.)

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Dynamic Programming using BDDs

Algorithm design choices

Early Decision Method (EDM)

- Information is incorporated in introduction nodes
- Comparable to “classical” table-based implementations
- `Unsatisfiable` instances: Conflicts are detected earlier

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Late Decision Method (LDM)

- BDD manipulation is delayed until removal of vertices
- Typically yields smaller BDDs and less computational effort
- Particularly useful for "complicated" algorithms
- Usually more concise algorithm specification

3-Colorability

Late Decision Method

$$\mathcal{B}^{l}_{n} = \top$$

3-Colorability

Late Decision Method

$\mathcal{B}_n^l = \top \qquad \mathcal{B}_n^i = \mathcal{B}_{n'}$

3-Colorability

Late Decision Method

$$\mathcal{B}_n^l = \top \qquad \mathcal{B}_n^i = \mathcal{B}_{n'} \qquad \mathcal{B}_n^j = \mathcal{B}_{n'} \land \mathcal{B}_{n''}$$

3-Colorability

Late Decision Method

$$\mathcal{B}^l_n = \top \qquad \mathcal{B}^i_n = \mathcal{B}_{n'} \qquad \mathcal{B}^j_n = \mathcal{B}_{n'} \wedge \mathcal{B}_{n''}$$

$$\begin{aligned}\mathcal{B}^r_n = &\Big(\mathcal{B}_{n'}[r_u/\top, g_u/\bot, b_u/\bot] \wedge \bigwedge_{\{x,u\}\in E_{n'}} \neg r_x\Big) \vee \\ &\Big(\mathcal{B}_{n'}[r_u/\bot, g_u/\top, b_u/\bot] \wedge \bigwedge_{\{x,u\}\in E_{n'}} \neg g_x\Big) \vee \\ &\Big(\mathcal{B}_{n'}[r_u/\bot, g_u/\bot, b_u/\top] \wedge \bigwedge_{\{x,u\}\in E_{n'}} \neg b_x\Big)\end{aligned}$$

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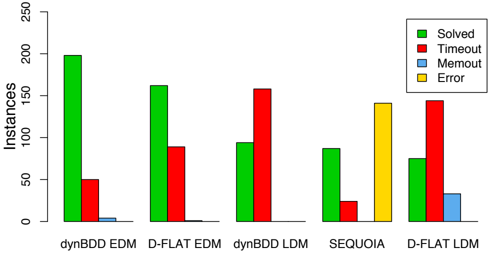
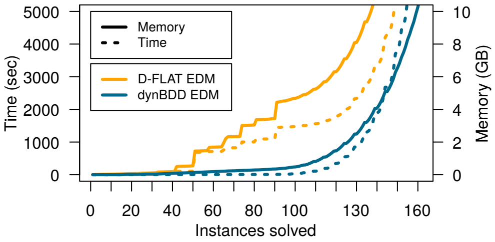
3-COLORABILITY

Preliminary experimental results





Related work

Practical realizations for DP on TDs

- Some problem-specific implementations (e.g. graph optimization, argumentation, . . .)
- SEQUOIA (2011): Takes MSO formula and does DP internally
- D-FLAT (2012): Specify algorithm for particular problem in ASP

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Further related approaches

- Branch and Bound on TDs [Allouche et al., 2015]
- Trees-of-BDDs [Fargier and Marquis, 2009]
- Optimization with decision diagrams [Bergman et al., 2015]

Conclusion

Current results

- Feasible for problems that are fpt w.r.t. tree-width w
  - Size of BDDs bounded by $\mathcal{O}(2^{w \cdot c})$
- So far, NP-complete problems were considered:
  - 3-Colorability: only variables with *fixed* truth value
  - Dominating Set variant: variables with *changing* truth value
  - Hamiltonian Cycle: handle *connectedness* in DP algorithm
- Development and study of design patterns EDM and LDM

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Future work

- Consider problems harder than NP (via *sets of BDDs*)
- Optimization problems (use alternatives to BDDs)
- Support for high-level algorithm specification
- Visualization and debugging support for algorithm development

http://dbai.tuwien.ac.at/proj/decodyn/dynbdd

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