Infinitary Equilibrium Logic and Strong Equivalence

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Overview

Strong equivalence of logic programs is an important concept in the theory of answer set programming.

Equilibrium logic was used to show that propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there (Pearce 1997; Lifschitz, Pearce, and Valverde, 2001; Ferraris 2005).

Infinitary propositional formulas have been used to define a precise semantics for a large subset of the ASP input language of GRINGO, called AG (Gebser, H., Kaminski, Lifschitz, and Schaub, ICLP'15).

We extend equilibrium logic to infinitary propositional formulas, define and axiomatize an infinitary counterpart to the logic of here-and-there, and show that the theorem on strong equivalence holds in the infinitary case as well.

Motivation: Semantics of Aggregate Expressions

The aggregate expression is an example of a construct that has been added to ASP input languages but is not covered by the original semantics.

Example: The expression

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#count{X:p(X)} = 0
```

intuitively says that the cardinality of the set of all X such that $p(X)$ holds is 0.

If there are infinitely many possible values for X the meaning of this expression cannot be represented using a propositional formula.

In AG, the meaning of aggregate expressions is captured using an infinitary propositional formula. The definition is based on the semantics for propositional aggregates due to Ferraris (2005).

Review:
Strong Equivalence and Equilibrium Logic

Strong Equivalence

About sets $\mathcal{H}_1$, $\mathcal{H}_2$ of formulas we say that they are *strongly equivalent* to each other if, for every set $\mathcal{H}$ of formulas, the sets $\mathcal{H}_1 \cup \mathcal{H}$ and $\mathcal{H}_2 \cup \mathcal{H}$ have the same stable models.

Example. The sets

$$\mathcal{H}_1 = \{p \vee q\}$$

$$\mathcal{H}_2 = \{\neg q \rightarrow p, \;\; \neg p \rightarrow q\}$$

are not strongly equivalent. If we add the set $\{p \rightarrow q, \;\; q \rightarrow p\}$ the resulting sets will have different stable models.

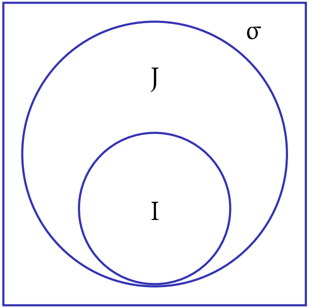
If we add $\{\neg(p \wedge q)\}$ to both $\mathcal{H}_1$ and $\mathcal{H}_2$, the resulting sets will be strongly equivalent.

Logic of Here-and-There

An *interpretation* is a subset of σ. An *HT-interpretation* is a pair $\langle I, J \rangle$ of subsets of σ such that $I \subseteq J$. Intuitively, each atom A is assigned one of three possible "truth values" : either $A \in I$, or $A \in J \setminus I$, or $A \notin J$.

The satisfaction relation between an HT-interpretation and a formula is defined recursively, as follows:

- For every atom A from σ, $\langle I, J \rangle \models A$ if $A \in I$.
- $\langle I, J \rangle \models F \wedge G$ if $\langle I, J \rangle \models F$ and $\langle I, J \rangle \models G$.
- $\langle I, J \rangle \models F \vee G$ if $\langle I, J \rangle \models F$ or $\langle I, J \rangle \models G$.
- $\langle I, J \rangle \models F \rightarrow G$ if
  - (i) $\langle I, J \rangle \not\models F$ or $\langle I, J \rangle \models G$, and
  - (ii) $J \models F \rightarrow G$.



Satisfying HT-interpretations are called *HT-models*.

Axiomatizing the Logic of Here-and-There

The first axiomatization was given without proof by Jan Łukasiewicz (1941): add the axiom schema

$$(\neg F \rightarrow G) \rightarrow (((G \rightarrow F) \rightarrow G) \rightarrow G))$$

to intuitionistic logic. This axiom schema was rediscovered and proved correct by Ivo Thomas (1962).

Toshio Umezawa (1959) observed that formulas of the form

$$F \vee (F \rightarrow G) \vee \neg G$$

are sound in HT.

Tsutomu Hosoi (1966) proved that HT can be axiomatized by adding this as an axiom schema to intuitionistic logic.

Characterizing Strong Equivalence

Strong equivalence is characterized by the logic of here-and-there.

Theorem (Lifschitz, Pearce, and Valverde, 2001; Ferraris 2005)

For any sets $\mathcal{H}_1$, $\mathcal{H}_2$ of formulas,

$\mathcal{H}_1$ is strongly equivalent to $\mathcal{H}_2$

iff

sets $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same HT-models.

The if-part allows us to establish strong equivalence by reasoning about HT-models, that is, by reasoning in intuitionistic logic + Hosoi's axiom.

The only-if-part tells us that, in priniciple, strong equivalence can always be proved using this method.

Equilibrium Models (Pearce, 1997)

The proof of the theorem on strong equivalence uses equilibrium models and the following result.

An HT-interpretation $\langle \mathrm{I}, \mathrm{J} \rangle$ is *total* if $\mathrm{I} = \mathrm{J}$.

An *equilibrium model* of a set $\mathcal{H}$ of formulas is a total HT-model $\langle \mathrm{J}, \mathrm{J} \rangle$ of $\mathcal{H}$ such that for every proper subset I of J, $\langle \mathrm{I}, \mathrm{J} \rangle$ is not an HT-model of $\mathcal{H}$.

Theorem

An interpretation J is a stable model of a set $\mathcal{H}$ of formulas iff $\langle \mathrm{J}, \mathrm{J} \rangle$ is an equilibrium model of $\mathcal{H}$.

Generalizing to Infinitary Formulas

Infinitary Formulas

For every nonnegative integer r, *(infinitary propositional) formulas (over signature σ) of rank* r are defined recursively, as follows:

- every atom from σ is a formula of rank 0,
- if $\mathcal{H}$ is a set of formulas, and r is the smallest nonnegative integer that is greater than the ranks of all elements of $\mathcal{H}$, then $\mathcal{H}^{\wedge}$ and $\mathcal{H}^{\vee}$ are formulas of rank r,
- if F and G are formulas, and r is the smallest nonnegative integer that is greater than the ranks of F and G, then $F \rightarrow G$ is a formula of rank r.

We write $\{F, G\}^{\wedge}$ as $F \wedge G$, and $\{F, G\}^{\vee}$ as $F \vee G$. The symbols $\top$ and $\bot$ are abbreviations for $\emptyset^{\wedge}$ and $\emptyset^{\vee}$ respectively; $\neg F$ stands for $F \rightarrow \bot$, and $F \leftrightarrow G$ stands for $(F \rightarrow G) \wedge (G \rightarrow F)$.

Stable Models of Infinitary Formulas (Truszczynski)

The definition of satisfaction between an interpretation and infinitary formula is a natural generalization of the finite case.

The *reduct* F^I of a formula F w.r.t. an interpretation I is defined to be:

- For $A \in \sigma$, $A^I = \bot$ if $I \not\models p$; otherwise $A^I = A$;
- $(\mathcal{H}^\wedge)^I = \{G^I \mid G \in \mathcal{H}\}^\wedge$;
- $(\mathcal{H}^\vee)^I = \{G^I \mid G \in \mathcal{H}\}^\vee$;
- $(G \to H)^I = \bot$ if $I \not\models G \to H$; otherwise $(G \to H)^I = G^I \to H^I$.

An interpretation I is a *stable model* of a set $\mathcal{H}$ of formulas if it is minimal among the interpretations satisfying F^I for all formulas F from $\mathcal{H}$.

Strong Equivalence of Infinitary Formulas

In the definition of satisfaction between an HT-interpretation and an infinitary formula, the clauses for conjunction and disjunction are:

- $\langle I, J \rangle \models \mathcal{H}^\wedge$ if for every formula F in $\mathcal{H}$, $\langle I, J \rangle \models F$;
- $\langle I, J \rangle \models \mathcal{H}^\vee$ if there is a formula F in $\mathcal{H}$ such that $\langle I, J \rangle \models F$.

The theorem on strong equivalence and its proof in terms of equilibrium models generalizes naturally to the infinitary case.

Theorem

For any sets $\mathcal{H}_1$, $\mathcal{H}_2$ of infinitary formulas,

$\mathcal{H}_1$ is strongly equivalent to $\mathcal{H}_2$

iff

sets $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same HT-models.

Reasoning about Aggregates in AG

Ferraris (2005) showed how formulas representing propositional monotone and antimonotone aggregates can be simplified using strongly equivalent transformations.

Extending this result to aggregates with global and local variables involves reasoning about strongly equivalent transformations of infinitary formulas.

In addition to the theorem on strong equivalence, we need an axiomatization of the infinitary logic of here-and-there.

Axiomatizing
the Infinitary Logic of Here-and-There

Axioms of HT^∞

The set of axioms in HT^∞ is a subset of the set of axioms introduced in the extended system of natural deduction from (H., Lifschitz, and Truszczynski, 2014):

$$F \Rightarrow F,$$

$$F \vee (F \to G) \vee \neg G,$$

and

$$\bigwedge_{\alpha \in A} \bigvee_{F \in \mathcal{H}_\alpha} F \to \bigvee_{(F_\alpha)_{\alpha \in A}} \bigwedge_{\alpha \in A} F_\alpha$$

for every non-empty family $(\mathcal{H}_\alpha)_{\alpha \in A}$ of sets of formulas such that its union is bounded; the disjunction in the consequent extends over all elements $(F_\alpha)_{\alpha \in A}$ of the Cartesian product of the family $(\mathcal{H}_\alpha)_{\alpha \in A}$.

Derivable objects are *(infinitary) sequents*—expressions of the form $\Gamma \Rightarrow F$, where F is an infinitary formula, and Γ is a *finite* set of infinitary formulas ("F under assumptions Γ").

$$(\wedge I)\ \frac{\Gamma \Rightarrow H \quad \text{for all } H \in \mathcal{H}}{\Gamma \Rightarrow \mathcal{H}^\wedge} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow \mathcal{H}^\wedge}{\Gamma \Rightarrow H} \quad (H \in \mathcal{H})$$

$$(\vee I)\ \frac{\Gamma \Rightarrow H}{\Gamma \Rightarrow \mathcal{H}^\vee} \quad (H \in \mathcal{H}) \qquad (\vee E)\ \frac{\Gamma \Rightarrow \mathcal{H}^\vee \quad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F}$$

$$(\rightarrow I)\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \qquad (\rightarrow E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \rightarrow G}{\Gamma, \Delta \Rightarrow G}$$

$$(W)\ \frac{\Gamma \Rightarrow F}{\Gamma, \Delta \Rightarrow F}$$

Proving Completeness of HT^∞

The set of theorems of HT^∞ is the smallest set that includes the axioms and is closed under the inference rules.

Theorem (Soundness and Completeness of HT^∞)

A formula is satisfied by all HT-interpretations iff it is a theorem of HT^∞.

Our proof of completeness is a generalization of a new proof of Hosoi's theorem.

The new proof is based on Kalmár's completeness proof for classical propositional logic.

Kalmár's Completeness Proof for Classical Logic

For any interpretation I, let M_I be the set $I \cup \{\neg A \mid A \in \sigma \setminus I\}$.

Lemma (Kalmár 1935)

For any formula F *and interpretation* I,

(i) *if* I *satisfies* F *then* F *is derivable from* M_I;

(ii) *if* I *does not satisfy* F *then* $\neg F$ *is derivable from* M_I.

Let F be a tautology containing atoms p, q only. By the lemma, F is derivable from each of the following formulas:

$$p \wedge q, \quad p \wedge \neg q, \quad \neg p \wedge q, \quad \neg p \wedge \neg q.$$

It remains to observe that by applying distributivity to

$$(p \vee \neg p) \wedge (q \vee \neg q)$$

we obtain the disjunction of these formulas.

Formulas Characterizing an HT-Interpretation

Recall that M_I stands for

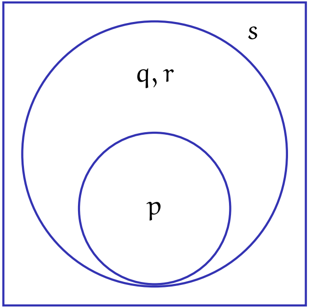
$$I \cup \{\neg A \mid A \in \sigma \setminus I\}.$$

For any HT-interpretation $\langle I, J \rangle$, let M_{IJ} be the set

$$I \;\cup\; \{\neg A \mid A \in \sigma \setminus J\} \;\cup\; \{\neg\neg A \mid A \in J \setminus I\} \;\cup\; \{A \rightarrow B \mid A, B \in J \setminus I\}\,.$$

Example: If $\langle I, J \rangle = \langle \{p\}, \{p, q, r\} \rangle$ and $\sigma = \{p, q, r, s\}$ then

$$M_{IJ} = \{\, p\,,\; \neg s\,,\; \neg\neg q, \neg\neg r\,,\; q \rightarrow r, r \rightarrow q, q \rightarrow q, r \rightarrow r\}\,.$$



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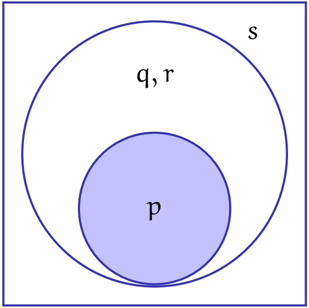

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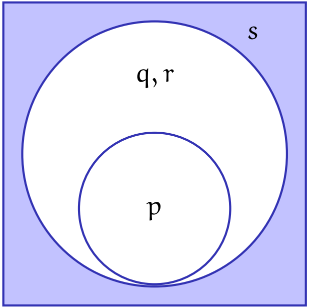

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Example: If $\langle I, J \rangle = \langle \{p\}, \{p, q, r\} \rangle$ and $\sigma = \{p, q, r, s\}$ then

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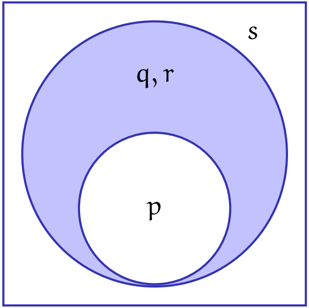
$$\neg s\,,$$
$$\neg\neg q, \neg\neg r\,,$$
$$q \to r, r \to q, q \to q, r \to r\}.$$



Main Lemma

In the statement of the lemma, derivability refers to derivability in Hosoi's deductive system.

Lemma

For any formula F *and HT-interpretation* $\langle I, J \rangle$,

(i) *if* $\langle I, J \rangle$ *satisfies* F *then* F *is derivable from* M_{IJ}*;*

(ii) *if* $\langle I, J \rangle$ *does not satisfy* F *but* J *satisfies* F *then for every atom* q *in* $J \setminus I$, $F \leftrightarrow q$ *is derivable from* M_{IJ}*;*

(iii) *if* J *does not satisfy* F *then* $\neg F$ *is derivable from* M_{IJ}*.*

Deriving Hosoi's Theorem

If F is HT-tautological then by part (i) of the main lemma it is derivable from M_{IJ} for any HT-interpretation $\langle I, J \rangle$.

By applying distributivity to the conjunction of the formulas $F \vee (F \rightarrow G) \vee \neg G$ for all literals F, G, we can prove:

Lemma

The disjunction of the formulas

$$\bigwedge_{F \in M_{IJ}} F$$

over all HT-interpretations $\langle I, J \rangle$ is provable in Hosoi's system.

It follows that F is provable.

This proof can be generalized to the infinitary case.

Conclusions

We have

- defined the infinitary version of the logic of here-and-there,
- defined its nonmonotonic counterpart—the infinitary version of equilibrium logic,
- verified that stable models of infinitary formulas can be characterized in terms of infinitary equilibrium logic,
- verified that infinitary propositional formulas are strongly equivalent to each other iff they are equivalent in the infinitary logic of here-and-there,
- found an axiomatization of that logic.

New work: (H., Lifschitz, and Michael, Finite Proofs for Infinitary Formulas, ASPOCP'15.)

The End

Questions?