Logic Programming with Graded Modality

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Motivation

- Express modal concepts: “at least as many as” and “at most as many as” naturally.

- Find natural and intuitive semantics for Epistemic Specification

Combine the basic idea in Graded Modal Logic and ASP!
Logic Programs with Graded Modality

• Syntax
  – Rule \( l_1 \) or \( \ldots \) or \( l_k \) :- \( e_1, \ldots, e_m \), \( s_1, \ldots, s_n \). \( k \geq 0, m \geq 0, n \geq 0 \)
  – Literals
    • Extended literal \( e \)
      » Objective literals: \( l \)
      » Default literals: \( \text{not } l \)
    • Subject literals:
      » With upper bound: \( M_{[lb:ub]} e \)
      » Without upper bound: \( M_{[lb:]} e \)

\( M_{[lb:ub]} e \) intuitively means: it is known that the number of belief sets where \( e \) is true is between \( lb \) and \( ub \).
Safe Rule:

A rule is **SAFE** if each variable in it appears in the positive body of the rule.

- The positive body of a rule $r$ is composed of the extended literals containing no **not** in its body.
Semantics (ground program):

Let $W$ be a non-empty collection of consistent sets of ground objective literals, $w \in W$,

- $<W, w> \models l$ if $l \in w$
- $<W, w> \models \text{not } l$ if $l \notin w$
- $<W, w> \models M_{[lb:ub]}e$ if $lb \leq |\{w \in W| <W, w> \models e\}| \leq ub$
- $<W, w> \models M_{[lb:]}e$ if $|\{w \in W| <W, w> \models e\}| \geq lb$

Then, for a rule $r$ in $\Pi$, $<W, w> \models r$ if

- $\exists l \in \text{head}(r): <W, w> \models l$
- $\exists t \in \text{body}(r): <W, w> \not\models t$.

$W$ is a model of a program $\Pi$, if $\forall r \in \Pi$, and $\forall w \in W$,

$<W, w> \models r$
A Special Model: World View I

  - Believe in the head of a rule if you believe in its body (*Satisfiability principle*)
  - Do not believe in contractions (*Consistency principle*)
  - Believe nothing you are not forced to believe (*Rationality principle*)
World View will be defined on

- the principles (From ASP), and

- the above satisfaction notion for literals (From GML), and rules (Satisfiability principle in ASP).
A Special Model: World View II

• Part I. For a disjunctive logic program, its world view is the non-empty set of all its answer sets.
A Special Model: World View III

• Part II. For an arbitrary $LPGM$ program $\Pi$, a non-empty set $W$ is its world view if $W$ is the world view of a DLP $\Pi^W$ which is a reduct of $\Pi$ by the following laws.
A Special Model: World View IV

\( \Pi^W \)

1. Removing from all rules containing subjective literals not satisfied by \( W \).

2. Replacing all other occurrences of subjective literals of the form \( M[1b; ub] \ l \) or \( M[1b; ] \ l \) where \( lb=|W| \) by \( l \).

3. Removing all other occurrences of subjective literals of the form \( M[1b; ub] \ not \ l \) or \( M[1b; ] \ not \ l \) where \( lb=|W| \).

4. Replacing all other occurrences of subjective literals of the form \( K[0; 0] \ e \) by \( e^{not} \). (\( e^{not} \) is \( not \ l \) if \( e \) is \( l \), and \( e^{not} \) is \( l \) if \( e \) is \( not \ l \))

5. Replacing other occurrences of subjective literals of the form \( M\omega \ e \) by \( e \) and \( e^{not} \) respectively, that is, two rules should be created, one in which \( M\omega \ e \) is replaced by \( e \) and one in which \( M\omega \ l \) is replaced by \( e^{not} \).
A Special Model: World View V

• Example $\Pi$:

\[ l : - M_{[1:]} l. \]

For $W=\{\{\}\}$, $\Pi^W$ is empty and has a world view $\{\{\}\}$, therefore, $\{\{\}\}$ is a world view of $\Pi$.

For $W=\{\{l\}\}$, $\Pi^W$ has one rule: $l: - l.$, and thus has a world view $\{\{\}\}$. Hence, $\{\{l\}\}$ is not a world view of $\Pi$. 
A Special Model: World View VI

The reduct is natural and intuitive.

• **Law 1**: Removing from all rules containing subjective literals not satisfied by $W$.

• **Interpretation**: The law directly comes from the notion of Rule Satisfiability and Rationality Principle in answer set programming which means if a rules body cannot be satisfied (believed in), the rule will contribute nothing;
A Special Model: World View VII

The reduct is natural and intuitive.

• **Law 2**: Replacing all other occurrences of subjective literals of the form \(M_{[1b:ub]} l\) or \(M_{[1b:]} l\) where \(lb=|W|\) by \(l\).

• **Interpretation**: if it is known that there are at least \(lb\) number of belief sets where \(l\) is true and there are totally \(lb\) belief sets in \(W\), then, by the meaning of the gradation based on counting, \(l\) must be believed with regard to each belief set in \(W\). Then, by the **Rationality Principle**, you are forced to believe \(l\) with regard to each belief set. Hence, \(M_{[1b:ub]} l\) or \(M_{[1b:]} l\) should be replaced by \(l\) (instead of being removed) to avoid self-support;
The reduct is natural and intuitive.

- **Law 3**: Removing all other occurrences of subjective literals of the form $\mathcal{M}_{[1b: ub]} \text{not } l$ or $\mathcal{M}_{[1b:]} \text{not } l$ where $lb = |W|$.

- **Interpretation**: If it is known that there are at least $lb$ number of belief sets where $l$ is not true and there are totally $lb$ belief sets in $W$, then, $\text{not } l$ must be believed with regard to each belief set in $W$. Then, removing $\mathcal{M}_{[1b: ub]} \text{not } l$ or $\mathcal{M}_{[1b:]} \text{not } l$ in a rule will not affect the satisfiability of the rule.
A Special Model: World View VIII

The reduct is natural and intuitive.

• **Law 4**: Replacing all other occurrences of subjective literals of the form $M_{[0:0]} e$ by $e_{not}\ldots$

• **Interpretation**: if it is known that $e$ is not believed with regard to each belief set in $W$, then we are forced to believe $e_{not}$ with regard to each belief set in $W$;
The reduct is natural and intuitive.

- **Law 5**: Replacing other occurrences of subjective literals of the form $M \circ e$ by $e$ and $e^{not}$ respectively.
- **Interpretation**: The last law states that, if it is known that there are at least $1b$ number of belief sets where $e$ is believed, and the number of belief sets in $W$ is strict greater than $1b$, then $e$ may be believed or may not be believed with regard to a belief set in $W$. 
• Example using Law 5.

\[ l :\overline{-} M_{[0:]}. \]

For \( W=\{\} \) (\(|W|=1\)), \( \prod^W \) is:

\[ l:\overline{-} l. \]

\[ l:\overline{-} \text{not } l. \]
Relation to Epistemic Specification

Latest *Epistemic Specification* (2014 version): \( \text{ASP}^{KM} \)

where \( K, M, \neg K, \neg M \) are used to extend \( \text{ASP} \).
Theorem. An ASP\textsuperscript{KM} Program can be represented as a LPGM program by

\begin{align*}
Kl & \Rightarrow M_{[0:0]} not \ l \\
Ml & \Rightarrow M_{[1:]}, l \ and \ not \ not \ l \ respectively. \\
not \ Kl & \Rightarrow M_{[1:]}, not \ l \ and \ not \ l \ respectively \\
not \ Ml & \Rightarrow M_{[0:0]} l
\end{align*}
A Sound and Complete Algorithm for Computing World Views

Algorithm 1 LPGMSolver.

Input:
\( \Pi \): A LPGM;

Output:
All world views of \( \Pi \);

1: \( n = \max \{ lb | M_{[lb:ub]} e \text{ or } M_{[lb]} e \text{ in } \Pi \} \) \{computes the maximal \( lb \) of subjective literals in \( \Pi \)\}
2: \( WV = \emptyset \)
3: for every natural number \( 1 \leq k \leq n \) do
4: \( WV_k = \text{WViSolver}(\Pi, k) \) \{computes all world views of size \( k \) for \( \Pi \)\}
5: \( WV = WV \cup WV_k \)
6: end for
7: \( WV_{>n} = \text{WViSolver}(\Pi, n) \) \{computes all world views of size strict greater than \( n \) for \( \Pi \)\}
8: \( WV = WV \cup WV_{>n} \)
9: output \( WV \)
An Algorithm for Computing World Views

WViSolver and WVgiSolver

A GI-log Program

Translated into an ASP program

Grounding and computing answer sets using DLV/Clingo

Group answer sets and test if a group is a world view

Generate

Test
Complexity of the Algorithm

• The algorithm is in PSPACE and $O(2^{3|\mathcal{L}|})$.

• A LPGM solver will be issued in

http://cse.seu.edu.cn/people/seu_zzzz/
Applications: A Case Study I

• **N-critical edges problem.** Given a directed graph $G = (V, E)$ where $V$ is the set of vertices of $G$ and $E$ is the set of edges of $G$, find the set of all edges that belong to $n$ or more hamiltonian cycles in $G$. 
Formalizing Hamiltonian cycles

\(\text{inhc}(X, Y) \text{ or } \neg\text{inhc}(X, Y) \leftarrow \text{edge}(X, Y).\)
\(\leftarrow \text{inhc}(X_1, Y_1), \text{inhc}(X_2, Y_1), X_1 \neq X_2.\)
\(\leftarrow \text{inhc}(X_1, Y_1), \text{inhc}(X_1, Y_2), Y_1 \neq Y_2.\)

\(\text{reachable}(X, X) \leftarrow \text{vertex}(X).\)
\(\text{reachable}(X, Y) \leftarrow \text{inhc}(X, Z), \text{reachable}(Z, Y).\)
\(\leftarrow \text{vertex}(X), \text{vertex}(Y), \text{not reachable}(X, Y).\)

Defining N-critical edges

\(\text{ncritical}(X, Y) \leftarrow M[n:]\text{inhc}(X, Y), \text{edge}(X, Y).\)
Applications: A Case Study II

• **N-exclusive paths problem.** Given a directed graph $G = (V, E)$ where $V$ is the set of vertices of $G$ and $E$ is the set of edges of $G$, decide whether there are $n$ number of $m$-exclusive paths from a vertex $a$ to another vertex $b$, that is, decide whether there are $n$ or more paths between $a$ and $b$, and there are no edge belonging to $m$ or more of the paths.
Formalizing the definition of $Path(a, b)$

\[
\text{inpath}(X, Y) \text{ or } \neg \text{inpath}(X, Y) \leftarrow \text{edge}(X, Y).
\]
\[
\leftarrow \text{inpath}(X1, Y1), \text{inpath}(X2, Y1), X1 \neq X2.
\]
\[
\leftarrow \text{inpath}(X1, Y1), \text{inpath}(X1, Y2), Y1 \neq Y2.
\]
\[
\text{reachable}(X, X) \leftarrow \text{vertex}(X).
\]
\[
\text{reachable}(X, Y) \leftarrow \text{inpath}(X, Z), \text{reachable}(Z, Y).
\]
\[
\text{path} \leftarrow \text{reachable}(a, b).
\]
\[
\leftarrow \text{not path}.
\]

Formalizing $n$ or more paths

\[
\text{npath} \leftarrow M_{[n:]\text{path}}.
\]
\[
\leftarrow \text{not npath}.
\]

Formalizing $m$-exclusive

\[
\leftarrow M_{[m:]\text{inpath}(X, Y), \text{edge}(X, Y)}.
\]
Conclusion

● LPGM language is a new way of reasoning with Negation as Failure and Modality together.

● LPGM semantics/reduct is intuitive and based on the principles of ASP.

● The application of LPGM seems potential.
Future Work

● Mathematical Properties.

● Methodologies for modeling with LPGM. E.g. Contextual Reasoning, information fusion etc.

● Other graded modalities
A new result can be found in ASPOCP 2015. “logic programming with graded introspection”
Thank You!