Al Direttore
del Dipartimento di Matematica e Informatica
dell' Università della Calabria

Oggetto: Attivazione procedura conferma in ruolo

Il sottoscritto Vittorio Colao, Ricercatore Universitario (S.S.D. Mat/05) presso il Dipartimento di Matematica e Informatica dell'Università della Calabria dal 24/02/2012, avendo maturato il triennio di prova utile per la conferma in ruolo in data 24/02/2015,

CHIEDE

al Consiglio di Dipartimento di deliberare sulla propria attività scientifica e didattica nel periodo interessato ai fini della **attivazione della procedura per la valutazione ed eventuale conferma in ruolo**. Allega alla presente una relazione sulle attività svolte unitamente a titoli e pubblicazioni.

Arcavacata di Rende, 20 Marzo 2015

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Relazione sull'Attività Scientifica e Didattica

Svolta dal 24 Febbraio 2012 al 24 Febbraio 2015

dal Dr. Vittorio Colao,

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Attività didattica.

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Titolare del corso di Analisi Matematica 4, Laurea di triennale in Matematica.

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Titolare del corso di Analisi Matematica 2, Laurea di triennale in Matematica e Laurea di triennale in Fisica.
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Titolare del corso di Analisi Matematica II, Laurea di triennale in Fisica.
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Relatore per n.2 tesi di Laurea di triennale in Matematica.
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Titolare del corso di Analisi Matematica II, Laurea di triennale in Fisica.
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Altre attività didattiche.

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Attività Scientifica.

Pubblicazioni.

I seguenti articoli sono stati accettati per la pubblicazione tra il 24 Febbraio 2012 e il 24 Febbraio 2015:

[**A**] V. Colao, G. Marino, D.R. Sahu, *A general inexact iterative method for monotone operators, equilibrium problems and fixed point problems of semigroups in Hilbert spaces*, **Fixed Point Theory and Applications** 2012 (1), 1-19 (2012).

[**B**] D.R. Sahu, V. Colao, G. Marino, *Strong convergence theorems for approximating common fixed points of families of nonexpansive mappings and applications*, **Journal of Global Optimization** 56 (4), 1631-1651 (2013).

[**C**] D. R. Sahu, V. Colao and G. Marino, *On the convergence of approximants of pseudo-contractive semigroups in Banach spaces*, **Journal of Nonlinear and Convex Analysis**, 15 (3), 547-556 (2014).

[**D**] V. Colao, L. Muglia, H.-K. Xu, *Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay*, **Annali di Matematica Pura e Applicata**, http://dx.doi.org/10.1007/s10231-015-0484-0 (2014).

[**E**] V. Colao, L. Muglia, *A hierarchical approach to fixed point problems for uniformly asymptotically regular sequences*, **Journal of Nonlinear and Convex Analysis**, To appear (2015).

Alla data del 24 Febbraio 2015, il seguente articolo, sottomesso alla rivista **Fixed Point Theory and Applications**, era in fase di revisione perchè l'Editore aveva comunicato la necessità di correzioni minori:

[F] V. Colao, G. Marino, *Krasnoselskii-Mann method for non-self mappings.*

Descrizione dell'attività di ricerca.

La mia attività di ricerca si svolge nell'ambito della Teoria dei punti fissi. In particolare, i problemi da me affrontati nel corso del triennio riguardano principalmente la convergenza di metodi iterativi per l'approssimazione di punti fissi in spazi di Banach e l'esistenza di soluzioni per equazioni differenziali impulsive con ritardo.

Metodi iterativi. Nel lavori [A], [B] e [C], si affronta lo studio di alcuni algoritmi iterativi convergenti a punti fissi comuni di famiglie di operatori soddisfacenti talune proprietà.

In particolare, diremo che una famiglia di operatori $\{T(t) : t \geq 0\}$, definiti su uno spazio di Banach X, soddisfa la proprietà $(\mathcal{A})$ rispetto ad una famiglia $\{S(t) : t \geq 0\}$ se per ogni rete $\{x_s\}$ per cui si abbia $x_s - S(s)x_s \to 0$ quando $s \to \infty$, ne segue che $\lim_{s\to\infty} x_s - T(t)x_s = 0$ per ogni $t > 0$. In [A], sono espresse alcune costruzioni abbastanza generali di famiglie di operatori soddisfacenti tale proprietà. Sempre in [A] e dati uno spazio di Hilbert H e una famiglia $\mathcal{T} = \{T(t) : t \geq 0\}$ di operatori non espansivi, da H in se e soddisfacente la proprietà illustrata, si prova la convergenza forte di uno schema implicito all'unico punto fisso x^*, comune agli elementi della famiglia $\mathcal{T}$, che soddisfi la diseguaglianza variazionale

$$\langle(\gamma f - A)x^*, p - x^*\rangle \leq 0 \quad \forall p \in Fix(\mathcal{T}), \tag{1}$$

dove γ è una opportuna costante, $f : H \to H$ una contrazione, A un operatore lineare fortemente positivo e dove $Fix(\mathcal{T})$ denota l'insieme dei punti fissi comuni agli elementi della famiglia $\mathcal{T}$. Successivamente, si prova la convergenza forte della successione generata da uno schema esplicito inesatto alla stessa soluzione di (1).

Si puntualizza che la tecnica utilizzata permette di rilassare le ipotesi sulla successione $\{e_n\}$ degli errori, consentendo che essa soddisfi solo la condizione

$$\lim_{n\to\infty} \|e_n\| = 0$$

a fronte dell'ipotesi

$$\sum_{n=0}^{\infty} \|e_n\| < +\infty,$$

utilizzata in risultati precedenti (si veda, ad esempio [1] e [2]).

Procedendo sulla stessa linea del precedente lavoro e per la medesima categoria di operatori, in [B], si prova la convergenza forte di una successione generata da un metodo implicito e da un nuovo algoritmo esplicito inesatto, nel contesto degli spazi di Banach riflessivi e aventi norma uniformemente Gâteaux differenziabile.

In entrambi gli articoli appena citati, sono esposte alcune applicazioni dei risultati ottenuti a problemi inerenti l'approssimazione sia di zeri di operatori monotoni e accretivi per mezzo di algoritmi prossimali, introdotti da Martinet [3] e Rockafellar [1], che di soluzioni a problemi di equilibrio (Combettes et al. [4]). I risultati ottenuti in [A] e in [B] generalizzano alcune conclusioni contenute in lavori precedenti

([5],[6],[7] e [8]).

In [C] si affrontano alcuni algoritmi inesatti per semigruppi di funzioni Lipschitziane e pseudo contrattive e se ne prova la convergenza forte ad un punto fisso comune agli elementi del semigruppo considerato. Come conseguenza e nel contesto di alcuni spazi di Banach, si risponde affermativamente ad un problema posto da H.-K. Xu in [9].

Nel lavoro [E], si introduce uno schema iterativo gerarchico per una successione di funzioni soddisfacenti talune proprietà introdotte in [10] e in [11]. Quindi, si prova che la convergenza della successione generata dall'algoritmo è governata dal limite

$$\lim_{n\to\infty} \frac{\alpha_n \beta_n}{\mu_n},$$

dove $\{\alpha_n\}$, $\{\beta_n\}$ e $\{\mu_n\}$ sono successioni di coefficienti che intervengono nell'espressione dell'algoritmo stesso.

Sia H uno spazio di Hilbert e sia K un suo sottoinsieme chiuso e strettamente convesso. In [F], si propone un algoritmo di tipo Krasnoselskii-Mann per mappe non espansive $T : C \to H$ soddisfacenti la condizione

$$Tx \in I_K(x) \quad \forall x \in K,$$

dove $I_K(x) := \{x + c(u - x) : c \geq 1 \ e \ u \in K\}$.

In particolare, definita la funzione $h : K \to [0, 1]$ come $h(x) :=$

$\inf\{\lambda \in [0,1] : \lambda x + (1-\lambda)Tx \in K\}$, si considerano le iterazioni

$$\begin{cases} x_0 \in K, \\ \alpha_0 := \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} := \max\{\alpha_n, h(x_{n+1})\} \end{cases} \tag{2}$$

e si prova che tale algoritmo è ben definito. Successivamente, si mostra che la successione $\{x_n\}$ così generata converge debolmente a un punto fisso x^* dell'operatore T e che tale convergenza è forte se $x^* \in \partial K$.

L'algoritmo appena descritto non trova riscontro in letteratura poichè i coefficienti $\{\alpha_n\}$ non sono determinati a priori ma sono costruiti ad ogni passo, di modo che l'algoritmo stesso sia ben posto.

Equazioni differenziali impulsive con ritardo. In [D], si prende in esame un'equazione differenziale con ritardo, con impulsi non istantanei e della forma

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x(\sigma(t))), \quad q.o.\ t \in \bigcup_{i=0}^{N}(s_i, t_{i+1}] \\ \\ x(t) = \gamma_i(t, x(t)), \quad t \in \bigcup_{i=1}^{N}(t_i, s_i], \\ \\ x(t) = \phi(t), \qquad t \in [-r, 0], \quad x'(0) = \phi'(0) = \eta, \end{cases}$$

dove x mappa $[-r, +\infty)$ in $(\mathbb{R}^n, |\cdot|)$ (dove $|\cdot|$ non è necessariamente la norma euclidea), $\mathcal{T} := \{0 < t_1 < \ldots < t_N\} \subset [0, +\infty)\}$, $s_0 = 0$, $t_{N+1} := +\infty$, $s_i \in (t_i, t_{i+1})$ per ogni $i = 1, \ldots, N$ e A è una matrice reale $n \times n$.

A differenza del caso impulsivo classico, esempi di equazioni differenziali con impulsi non istantanei sono stati introdotti solo recentemente (si veda [12]) e studiati solo in intervalli chiusi e limitati.

Al fine di provare l'esistenza di soluzioni limitate su $[-r, +\infty)$, si dimostra un criterio di compattezza nello spazio delle funzioni continue a tratti e limitate. Tale criterio permette di applicare un teorema di punto fisso di tipo Krasnoselskii e introdotto in [13]. Un opportuno corollario prova che l'esistenza di soluzioni non necessariamente limitate è garantita da condizioni più deboli.

References

[1] R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (5) (1976) 877–898.

[2] H.-K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. (2) 66 (01) (2002) 240–256.

[3] B. Martinet, Brève communication. Régularisation d'inéquations variationnelles par approximations successives, ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique 4 (R3) (1970) 154–158.

[4] P. L. Combettes, S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (1) (2005) 117–136.

[5] G. Marino, H.-K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318 (1) (2006) 43–52.

[6] S. Plubtieng, R. Punpaeng, Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, Math. Comput. Modelling 48 (1) (2008) 279–286.

[7] N. Shioji, W. Takahashi, Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces, Nonlinear Anal. 34 (1) (1998) 87–99.

[8] T. Shimizu, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211 (1) (1997) 71–83.

[9] H.-K. Xu, A strong convergence theorem for contraction semigroups in Banach spaces, Bull. Austral. Math. Soc. 72 (03) (2005) 371–379.

[10] G. Marino, L. Muglia, On the auxiliary mappings generated by a family of mappings and solutions of variational inequalities problems, Optim. Lett. (2015) 263–282.

[11] G. Marino, L. Muglia, Y. Yao, The uniform asymptotical regularity of families of mappings and solutions of variational inequality problems, J. Nonlinear Convex Anal. 15 (3) (2014) 477–492.

[12] E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, Proc. Amer. Math. Soc. 141 (5) (2013) 1641–1649.

[13] J. Garcia-Falset, Existence of fixed points for the sum of two operators, Math. Nachr. 283 (12) (2010) 1736–1757.

Soggiorni presso altre Istituzioni Scientifiche.

Marzo 2012 - Visiting Professor presso National Sun Yat-sen University, Kaohsiung, Taiwan (30 giorni).
Dicembre 2014 - Visiting Professor presso National Sun Yat-sen University, Kaohsiung, Taiwan (18 giorni).

Convegni.

17 - 19 Settembre 2013 - Incontro Gruppo Italiano di Analisi Funzionale (G.I.A.F.) - Università di Genova (GE).
20 - 22 Dicembre 2013 - International Conference on Nonlinear Analysis and Optimization (ICNAO2013) - National Sun Yat-sen University, Kaohsiung, Taiwan.
15 Dicembre 2014 - International Symposium on Nonlinear Analysis and Optimization, I, National Sun Yat-sen University, Kaohsiung, Taiwan.

Talk e Seminari.

Compattezza in spazi di funzioni differenziabili su intervalli non limitati (Incontro G.I.A.F., 2013, Università di Genova, Genova).
Equilibrium problems in Hadamard manifolds (ICNAO2013, 2013, National Sun Yat-sen University, Kaohsiung, Taiwan).
Krasnoselskii-Mann iterations (International Symposium on Nonlinear Analysis and Optimization, I, 2014, National Sun Yat-sen University, Kaohsiung, Taiwan).

Altre Attività.

Reviewer per Zentralblatt MATH e per diverse riviste quali, ad esempio, Journal of Mathematical Analysis and Applications, Journal of Inequalities and Applications e Fixed Point Theory and Applications.

Arcavacata di Rende, 20 Marzo 2015

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A general inexact iterative method for monotone operators, equilibrium problems and fixed point problems of semigroups in Hilbert spaces

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Abstract

Let H be a real Hilbert space. Consider on H a nonexpansive family $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ with a common fixed point, a contraction f with the coefficient $0 < \alpha < 1$, and a strongly positive linear bounded self-adjoint operator A with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and that $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ is a family of nonexpansive self-mappings on H such that $F(\mathcal{T}) \subseteq F(\mathcal{S})$ and $\mathcal{T}$ has property $(\mathcal{A})$ with respect to the family $\mathcal{S}$. It is proved that the following schemes (one implicit and one inexact explicit):

$$x_t = b_t \gamma f(x_t) + (I - b_t A) S(t) x_t$$

and

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) S(t_n) x_n + e_n, \quad n \geq 0$$

converge strongly to a common fixed point $x^* \in F(\mathcal{T})$, where $F(\mathcal{T})$ denotes the set of common fixed point of the nonexpansive semigroup. The point x^* solves the variational in-equality $\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0$ for all $x \in F(\mathcal{T})$. Various applications to zeros of monotone operators, solutions of equilibrium problems, common fixed point problems of nonexpansive semigroup are also presented. The results presented in this article mainly improve the corresponding ones announced by many others.
2010 Mathematics Subject Classification: 47H09; 47J25.

Keywords: nonexpansive semigroup, common fixed point, contraction, variational inequality

1. Introduction

Let H be a real Hilbert space and T be a nonlinear mapping with the domain $D(T)$. A point $x \in D(T)$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in D(T): Tx = x\}$. Recall that T is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \forall x, y \in D(T).$$

Recall that a family $\mathcal{T} = \{T(s) : s \geq 0\}$ of mappings from H into itself is called a one-parameter nonexpansive semigroup if it satisfies the following conditions:

(i) $T(0)x = x$, $\forall x \in H$;

(ii) $T(s + t)x = T(s)T(t)x$, $\forall s, t \geq 0$ and $\forall x \in H$;

(iii) $||T(s)x - T(s)y|| \le ||x - y||$, $\forall s \ge 0$ and $\forall x, y \in H$;

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{T})$ the set of common fixed points of $\mathcal{T}$, that is, $F(\mathcal{T}) = \cap_{0 \le s \le \infty} F(T(s))$. For each $t > 0$ and $x \in C$, $\sigma_t(x)$ is the average defined by $\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\,ds$. It is known that $F(\mathcal{T})$ is closed and convex; see [1]. Let C be a nonempty closed and convex subset of H. One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping; see [2,3]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C,$$

where $u \in C$ is a fixed element. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of $\{x_t\}$ is as $t \to 0$, even T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved the following well-known strong convergence theorem.

Theorem B. *Let C be a closed convex bounded subset of a Hilbert space H and let T be a nonexpansive mapping on C. Fix $u \in C$ and define $z_t \in C$ as $z_t = tu + (1 - t)Tz_t$ for $t \in (0, 1)$. Then as $t \to 0$, $\{z_t\}$ converges strongly to an element of $F(T)$ nearest to u.*

As motivated by Theorem B, Halpern [4] considered the following explicit iteration:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{1.1}$$

and proved the following theorem.

Theorem H. *Let C be a closed convex bounded subset of a Hilbert space H and let T be a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0, 1]$ by $\alpha_n = n^{-\theta}$, $0 < \theta < 1$. Define a sequence $\{x_n\}$ by (1.2). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u.*

In 1977, Lions [5] improved the result of Halpern, still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of T where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}^2} = 0$.

It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = \frac{1}{n+1}$. This was overcome in 1992 by Wittmann [6], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Recall that a mapping $f : H \to H$ is an α-contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$||f(x) - f(y)|| \le \alpha||x - y||, \quad \forall x, y \in H.$$

Recall that an operator A is strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}||x||^2, \quad \forall x \in H.$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [7-13] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in D} \frac{1}{2} \langle Ax, x\rangle - \langle x, b\rangle, \tag{1.2}$$

where A is a linear bounded operator, D is the fixed point set of a nonexpansive mapping T and b is a given point in H. In [11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A) Tx_n + \alpha_n b, \quad n \geq 0,$$

strongly converges to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Marino and Xu [10] studied the following continuous scheme

$$x_t = t\gamma f(x_t) + (I - tA) Tx_t,$$

where f is an α-contraction on a real Hilbert space H, A is a bounded linear strongly positive operator and $\gamma > 0$ is a constant. They showed that $\{x_t\}$ strongly converges to a fixed point $\bar{x}$ of T. Also in [10] they introduced a general explicit iterative scheme by the viscosity approximation method:

$$x_n \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) Tx_n, \quad n \geq 0 \tag{1.3}$$

and proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a unique solution of the variational inequality:

$$\left\langle \left(A - \gamma f\right) x^*, x - x^*\right\rangle \geq 0, \quad \forall x \in F(T),$$

which is the optimality condition for the minimization problem:

$$\min_{x\in F(T)} \frac{1}{2} \langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

It is an interesting problem to study above (Browder's, Halpern's and so on) results with respect to the nonexpansive semigroup case. So far, only partial answers have been obtained. Recently, Plubtieng and Punpaeng [14] considered the iteration process $\{x_n\}$ generated by

$$x_0 \in H, \ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds, \ n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ with $\alpha_n + \beta_n < 1$ and $\{t_n\}$ is a positive real divergent sequence. They proved, under certain appropriate conditions on $\{\alpha_n\}$, that $\{x_n\}$ converges strongly to a common fixed point of one-parameter nonexpansive semigroup $\mathcal{T} = \{T(s) : s \geq 0\}$.

In this article, motivated by Li et al. [8], Marino and Xu [10], Plubtieng and Punpaeng [14], Cianciaruso et al. [15], Shioji and Takahashi [16] and Shimizu and Takahashi [17], we consider the following more general schemes (one implicit and one inexact explicit):

$$x_t = b_t \gamma f(x_t) + (I - b_t A) S(t) x_t$$

and

$$x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) S(t_n) x_n + e_n, \quad n \geq 0$$

where $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ is a family of arbitrary nonexpansive self-mappings on H with a common fixed point, $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ is a family of nonexpansive self-mappings on H such that $\mathcal{T}$ has property $(\mathcal{A})$ with respect to the family $\mathcal{S}$ and $F(\mathcal{T}) \subseteq F(\mathcal{S})$, $\gamma > 0$ is a constant, $f : H \to H$ is an α-contraction, A is a bounded linear strongly positive self-adjoint operator on H and $\{b_t\}$ is a net in (0, 1). Furthermore, by applying these results, we obtain iterative algorithms for zeros of monotone operators, equilibrium problems, and common fixed point problems of nonexpansive semigroups in real Hilbert spaces.

The results presented in this article improve and extend the corresponding results announced by Marino and Xu [10], Plubtieng and Punpaeng [14], Cianciaruso et al. [15], Shioji and Takahashi [16], and Shimizu and Takahashi [17]. We remark that our results are very similar to those of Li et al. [8]. However, it seems that can be a gap in the proofs of Li et al. results. Indeed, their semigroups and the contraction are self-mappings defined on a closed convex subset C of the Hilbert space H, while the strongly positive linear bounded operator is defined on H. So both the schemes involve not a convex combination, that this they are of interest only in the case $C = H$.

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the following section. Some of them are known; others are not hard to derive.

Lemma 2.1. (Shimizu and Takahashi [[17], Lemma 2]). *Let C be a nonempty closed convex bounded subset of a Hilbert space H, and $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$a strongly continuous semigroup of nonexpansive mappings from C into itself. Let $\sigma_t(x) := \frac{1}{t}\int_0^t T(s)x ds$. Then*

$$\lim_{t\to\infty} \sup_{x\in D} \|\sigma_t(x) - T(h)\,\sigma_t(x)\| = 0 \ \textit{for all}\ h > 0.$$

Lemma 2.2. ([[18], Corollary 5.6.4], [19]) *(Demiclosedness principle) Let H be a Hilbert space, C is a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping. Then $I - T$ is demiclosed, i.e., if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ strongly converges to y, then $(I - T)x = y$.*

Lemma 2.3. ([[18], Corollary 5.2.29]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to C$ a nonexpansive mapping. Then F(T) is closed and convex.*

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C (i.e., for $x \in H$, $P_C x$ is the only point in C* *such that* $||x - P_C x|| = \inf\{||x - z||: z \in C\}$*). Given* $x \in H$ *and* $z \in C$*. Then* $z = P_C x$ *if and only if there holds the relations:*

$$\langle x - z, y - z\rangle \leq 0, \ \forall y \in C. \tag{2.1}$$

Lemma 2.5. *Let* H *be a real Hilbert space,* $f: H \to H$ *an* α*-contraction, and* A *is a strongly positive linear bounded self-adjoint operator on* H *with the coefficient* $\bar{\gamma} > 0$*. Then, for* $0 < \gamma < \bar{\gamma}/\alpha$*,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\,\|x - y\|^2, \quad x, y \in H. \tag{2.2}$$

That is, $A - \gamma f$ *is strongly monotone with coefficient* $\bar{\gamma} - \alpha\gamma$*.*

Remark 2.6. *Taking* $\gamma = 1$ *and* $A = I$*, the identity mapping, we have the following inequality:*

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \alpha)\,||x - y||^2, \quad x, y \in H. \tag{2.3}$$

Furthermore, if f is a nonexpansive mapping in Remark 2.6, we have

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq 0, \quad x, y \in H. \tag{2.4}$$

Lemma 2.7.[10]. *Assume* A *is a strongly positive linear bounded self-adjoint operator on a real Hilbert space* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq ||A||^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.8. [12]. *Let* $\{\alpha_n\}$ *be a sequence of nonnegative real numbers satisfying the following condition:*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\sigma_n\}$ *is a sequence of real numbers such that*

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ *converges to zero.*

Let C be a nonempty subset of a Banach space X. Throughout this article, G denotes an unbounded set of $\mathbb{R}^+ := [0, \infty)$ such that $s + t \in G$ for all $s, t \in G$ (often $G = \mathbb{N}$or $\mathbb{R}^+$) and $\mathcal{B}(C)$ denotes collection of all bounded subsets of C. Let $\mathcal{T} = \{T_s : s \in G\}$ be a family of mappings from C into itself. Then:

(i) a sequence $\{x_n\}$ in C is said to be an approximate fixed point sequence of $\mathcal{T}$ if $\lim_{n\to\infty} \|x_n - T_{\mathcal{T}} x_n\| = 0$ for all $\tau \in G$,

(ii) $\mathcal{T} = \{T_s : s \in G\}$ is said to *uniformly asymptotically regular on* C (for short, u.a.r. on C) (see, [20]) if

$$\lim_{t\in G, t\to\infty} (\sup_{x\in\tilde{C}} ||T_t x - T_s T_t x||) = 0 \text{ for all } s \in G \text{ and } \tilde{C} \in \mathcal{B}(C).$$

A family $\mathcal{T} = \{T_s : s \in G\}$ satisfies property $(\mathcal{A})$ if the following holds:

each $\{x_s\}_{s\in G} \in \mathcal{B}(C)$ with $x_s - T_s x_s \to 0$ as $s \to \infty \Rightarrow x_s - T_t x_s \to \infty$ for all $t \in G$.

Remark 2.9. *If* $\mathcal{T}$ *be a singleton, i.e.,* $\mathcal{T} = \{T\}$*, or* $T_s = T$ *for all* s *in* G*, then* $\{T\}$ *always has property* $(\mathcal{A})$*.*

We further remark that the notion of uniform asymptotic regularity introduced by Edelstein and O'Brien [21] plays an important role for studying property $(\mathcal{A})$ of nonlinear Lipschitzian-type operators. Indeed, if $\mathcal{T} = \{T_s : s \in G\}$ is a nonexpansive semigroup and u.a.r., then $\mathcal{T}$ has property $(\mathcal{A})$. Indeed, for $\{y_s\} \in \mathcal{B}(C)$ and $t \in G$,

$$\begin{aligned}\|y_s - T_t y_s\| &\leq \|y_s - T_s y_s\| + \|T_s y_s - T_t T_s y_s\| + \|T_t T_s y_s - T_t y_s\| \\ &\leq 2\|y_s - T_s y_s\| + \sup_{y \in \{y_\gamma : \gamma \in G\}} \|T_s y - T_t T_s y\| \to 0 \text{ as } s \to \infty.\end{aligned}$$

We now introduce property $(\mathcal{A})$ of $\mathcal{T}$ with respect to the family $\mathcal{G}$.

Let C be a nonempty subset of a Banach space X and $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a family of mappings from C into itself with $\cap_{t>0} F(T(t)) \neq \emptyset$. Let $\mathcal{G} = \{G_t : t \in \mathbb{R}^+\}$ be a family of mappings from C into itself such that $\cap_{t>0} F(T(t)) \neq \cap_{t>0} F(G_t)$. We say the family $\mathcal{T} = \{T(s) : s \in G\}$ has property $(\mathcal{A})$ with respect to the family $\mathcal{G} = \{G_t : t \in \mathbb{R}^+\}$ if the following holds:

each $\{x_s\}_{s\in G} \in \mathcal{B}(C)$ with $x_s - G_s x_s \to 0$ as $s \to \infty \Rightarrow x_s - T(t)x_s \to 0$ as $s \to \infty$ for all $t > 0$.

Remark 2.10. *If a family $\mathcal{T} = \{T(s) : s \in G\}$has property $(\mathcal{A})$, then $\mathcal{T}$has property $(\mathcal{A})$with respect to itself.*

We now give some examples.

Example 2.11. *Let C be a nonempty closed convex subset of a Banach space X and T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. For each $t \in G$, and $b_t \in \mathbb{R}$with $0 < a \leq b_t \leq b < 1$, define $G_t : C \to C$ by*

$$G_t x = (1 - b_t)x + b_t Tx \text{ for all } x \in C.$$

Then T has property $(\mathcal{A})$with respect to family $\{G_t : t \in G\}$.

Proof. Let $\{x_t\}_{t\in G} \in \mathcal{B}(C)$ such that $\| x_t - G_t(x_t)\| \to 0$ as $t \to \infty$. Note that

$$\|x_t - Tx_t\| = b_t \|x_t - G_t(x_t)\|$$

and $0 < a \leq b_t \leq b < 1$ for all $t \in G$. Therefore, $\|x_t - Tx_t\| \to 0$ as $t \to \infty$. □

The following proposition shows that in a uniformly convex Banach space, nonexpansive semigroup $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ has property $(\mathcal{A})$ with respect to a nonexpansive semigroup $\{\sigma_t : t \in \mathbb{R}^+\} = \{\frac{1}{t}\int_0^t T(s)xds : t \in \mathbb{R}^+\}$.

Example 2.12. *Let D be a nonempty closed convex bounded subset of a Hilbert space H, and $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$be a strongly continuous semigroup of nonexpansive mappings from D into itself. For each $t > 0$, let $x_t \in D$ such that $\|x_t - \sigma_t(x_t)\| \to 0$ as $t \to \infty$. Then $\|x_t - T(\mathcal{T})x_t \to 0\|$as $t \to \infty$ for each $\tau > 0$.*

Proof. Let $\tau > 0$. Observe that

$$\begin{aligned}\|T(\tau)x_t - x_t\| &\leq \|T(\tau)x_t - T(\tau)\sigma_t(x_t)\| + \|T(\tau)\sigma_t(x_t) - \sigma_t(x_t)\| + \|\sigma_t(x_t) - x_t\| \\ &\leq 2\|x_t - \sigma_t(x_t)\| + \|T(\tau)\sigma_t(x_t) - \sigma_t(x_t)\| \\ &\leq 2\|x_t - \sigma_t(x_t)\| + \sup_{x\in D}\|T(\tau)\sigma_t(x) - \sigma_t(x)\|.\end{aligned}$$

By Lemma 2.1, we obtain that $\|x_t - T(\tau)x_t\| \to 0$ as $t \to \infty$ for each $\tau > 0$. □

3. Main results

Let H be a real Hilbert space and $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ a family of nonexpansive self-mappings on H with $F(\mathcal{S}) \neq \emptyset$. By Lemma 2.3, $F(\mathcal{S})$ is closed and convex. Let A be a strongly positive linear bounded self-adjoint operator of H into itself with coefficient $\bar{\gamma} > 0$ and $f : H \to H$ an α-contraction. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\{b_t : t > 0\}$ is a net in $(0, ||A||^{-1})$ such that $\lim_{t\to\infty} b_t = 0$. For each $t > 0$, the mapping $G_t : H \to H$ defined by

$$G_t x := b_t \gamma f(x) + (I - b_t A) S(t) x, \quad x \in H$$

is a contraction with Lipschitz constant $1 - b_t(\bar{\gamma} - \alpha\gamma)$. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}\|G_t x - G_t y\| &\le \|(1 - b_t A)\left(S(t)x - S(t)y\right)\| + \gamma b_t \|fx - fy\| \\ &\le (1 - b_t\bar{\gamma})\|x - y\| + \gamma b_t \alpha \|x - y\| \\ &= \left[1 - b_t(\bar{\gamma} - \alpha\gamma)\right]\|x - y\|.\end{aligned}$$

By the Banach contraction principle, G_t has a unique fixed point, denoted by, x_t in H, which uniquely solves the fixed point equation

$$x_t = b_t \gamma f(x_t) + (I - b_t A) S(t) x_t. \tag{3.1}$$

Lemma 3.1. *Let H be a real Hilbert space and $\mathcal{S} = \{S(t) : 0 \le t < \infty\}$ be a family of nonexpansive self-mappings on H such that $F(\mathcal{S}) \ne \emptyset$. Let $f : H \to H$ be an α-contraction, A is a strongly positive linear bounded self-adjoint operator of H into itself with coefficient $\bar{\gamma} > 0$. Let $\{b_t : t > 0\}$ be a net in $(0, ||A||^{-1})$ such that $\lim_{t\to\infty} b_t = 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and x_t is defined by (3.1). Then we have the following:*

(a) *There is a nonempty closed convex bounded subset D of H such that D is $S(t)$-invariant for each $t > 0$ and $\{x_t\}$ is in D.*

(b) $||x_t - S(t)x_t|| \to 0$ *as* $t \to \infty$.

Proof. (a) Taking $p \in F(\mathcal{S})$, we have

$$\begin{aligned}\|x_t - p\| &= \|b_t\gamma f(x_t) + (I - b_t A) S(t) x_t - p\| \\ &\le b_t \|\gamma f(x_t) - Ap\| + (1 - b_t\bar{\gamma})\|S(t)x_t - p\| \\ &\le b_t \|\gamma f(x_t) - Ap\| + (1 - b_t\bar{\gamma})\|x_t - p\| \\ &\le b_t\gamma \|f(x_t) - f(p)\| + b_t\|\gamma f(p) - Ap\| + (1 - b_t\bar{\gamma})\|x_t - p\| \\ &\le \left[1 - b_t(\bar{\gamma} - \gamma\alpha)\right]\|x_t - p\| + b_t\|\gamma f(p) - Ap\|.\end{aligned}$$

It follows that

$$\|x_t - p\| \le \frac{1}{\bar{\gamma} - \alpha\gamma}\|\gamma f(p) - Ap\|.$$

This implies that $\{x_t\}$ is bounded. Let D be the ball $B(p, r)$, centered in p and with radius $r = \frac{1}{\bar{\gamma}-\alpha\gamma}\|\gamma f(p) - Ap\|$, i.e., $D = \left\{w \in H : \|w - p\| \le \frac{1}{\bar{\gamma}-\alpha\gamma}\|\gamma f(p) - Ap\|\right\}$. Then $\{x_t\}$ is contained in set D. Moreover,

$$\begin{aligned}\|S(t)x_t - p\| &= \|S(t)x_t - S(t)p\| \\ &\le \|x_t - p\| \\ &\le \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|.\end{aligned}$$

Thus, D is a nonempty closed convex bounded subset of H and $S(t)$-invariant.

(b) The boundedness of $\{x_t\}$ implies that $\{fx_t\}$ and $\{AS(t)x_t\}$ are bounded. Thus, $\|x_t - S(t)x_t\| = b_t\|\gamma f(x_t) - AS(t)x_t\| \to 0$ as $t \to \infty$. □

We now establish our strong convergence theorems.

Theorem 3.2. *(Implicit scheme) Let H be a real Hilbert space H and $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ be a family of nonexpansive self-mappings on H such that*

$F(\mathcal{T}) \neq \emptyset$. Let $f : H \to H$ be an α-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Let $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ be a family of nonexpansive self-mappings on H such that $\mathcal{T}$ has property $(\mathcal{A})$ with respect to the family $\mathcal{S}$ and $F(\mathcal{T}) \subseteq F(\mathcal{S})$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and that $\{b_t : t > 0\}$ is a net in $(0, ||A||^{-1})$ such that $\lim_{t\to\infty} b_t = 0$. Then $\{x_t\}$ defined by (3.1) strongly converges as $t \to \infty$ to $x^* \in F(\mathcal{T})$, where $x^* = P_{F(\mathcal{T})}(I - A + \gamma f)$ is a solution of the following variational inequality

$$\left\langle \left(\gamma f - A\right) x^*, p - x^* \right\rangle \leq 0, \quad \forall p \in F(\mathcal{T}). \tag{3.2}$$

Proof. The uniqueness of the solution of the variational inequality (3.2) is a consequence of the strong monotonicity of $A-\gamma f$ (Lemma 2.5). Next, we shall use $x^* \in F(\mathcal{T})$ to denote the unique solution of (3.2). To prove that $x_t \to x^*$ $(t \to \infty)$, we write, for a given $p \in F(\mathcal{T})$,

$$x_t - p = b_t \left(\gamma f(x_t) - Ap\right) + (I - b_t A) \left(S(t) x_t - p\right).$$

Using $x_t - p$ to make inner product, we obtain that

$$\begin{aligned} \|x_t - p\|^2 &= \left\langle (I - b_t A) \left(S(t) x_t - p\right), x_t - p \right\rangle + b_t \left\langle \gamma f(x_t) - Ap, x_t - p \right\rangle \\ &\leq (1 - b_t \bar{\gamma}) \|x_t - p\|^2 + b_t \left\langle \gamma f(x_t) - Ap, x_t - p \right\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_t - p\|^2 &\leq \frac{1}{\bar{\gamma}} \left(\gamma \left\langle f(x_t) - f(p), x_t - p \right\rangle + \left\langle \gamma f(p) - Ap, x_t - p \right\rangle\right) \\ &\leq \frac{\gamma\alpha}{\bar{\gamma}} \|x_t - p\|^2 + \frac{1}{\bar{\gamma}} \left\langle \gamma f(p) - Ap, x_t - p \right\rangle, \end{aligned}$$

which yields that

$$\|x_t - p\|^2 \leq \frac{1}{\bar{\gamma} - \alpha\gamma} \left\langle \gamma f(p) - Ap, x_t - p \right\rangle. \tag{3.3}$$

Since H is a Hilbert space and $\{x_t\}$ is bounded as $t \to \infty$, there exists a sequence $\{t_n\}$ in $(0, \infty)$ such that $t_n \to \infty$ and $x_{t_n} \rightharpoonup \bar{x} \in H$. By Lemma 3.1(b), we have $||x_t - S(t)x_t|| \to 0$ as $t \to \infty$. Since $\mathcal{T}$ has property $(\mathcal{A})$ with respect to the family $\mathcal{S}$, it follows that $x_t - T(\mathcal{T})x_t \to 0$ as $t \to \infty$ for all $\tau > 0$. Hence, by Lemma 2.2, $\bar{x} \in F(\mathcal{T}) \subseteq F(\mathcal{S})$. By (3.3), we see $x_{t_n} \to \bar{x}$. We next prove that $\bar{x}$ solves the variational inequality (3.2). From (3.1), we arrive at

$$\left(A - \gamma f\right) x_t = -\frac{1}{t} (I - tA) \left[x_t - S(t) x_t\right].$$

For $p \in F(\mathcal{T})$, it follows from (2.4) that

$$\begin{aligned} &\left\langle \left(A - \gamma f\right) x_t, x_t - p \right\rangle = -\frac{1}{t} \left\langle (I - tA) \left[x_t - S(t) x_t\right], x_t - p \right\rangle \\ &= -\frac{1}{t} \left\langle \left[(I - S(t)) x_t - (I - S(t)) p\right], x_t - p \right\rangle + \left\langle A(I - S(t))x_t, x_t - p \right\rangle \\ &\leq \left\langle A(I - S(t)) x_t, x_t - p \right\rangle. \end{aligned}$$

Since $x_{t_n} \rightharpoonup \bar{x}$, we obtain

$$\left\langle \left(A - \gamma f\right)\bar{x}, \bar{x} - p\right\rangle \leq 0,$$

i.e., $\bar{x}$ satisfies the variational inequality (3.2). By the uniqueness, it follows $\bar{x} = x^*$.

In a summary, we have shown that each cluster point of $\{x_t\}$ (as $t \to \infty$) equals x^*. Therefore, $x_t \to x^*$ as $t \to \infty$. The variational inequality (3.2) can be rewritten as

$$\left\langle \left[\left(I - A + \gamma f\right) x^*\right] - x^*, x^* - p\right\rangle \geq 0, \quad p \in F(\mathcal{T}).$$

This, by Lemma 2.4, is equivalent to

$$P_{F(\mathcal{T})}\left(I - A + \gamma f\right) x^* = x^*.$$

This completes the proof. □

Theorem 3.3. *(Inexact explicit scheme) Let H be a real Hilbert space H and* $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$*be a family of nonexpansive self-mappings on H such that* $F(\mathcal{T}) \neq \emptyset$, $f : H \to H$ *be an* α*-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient* $\bar{\gamma} > 0$*. Let* $\{t_n\}$ *be a positive real divergent sequence and let* $\Gamma = \left\{S_{t_n} : n \in \mathbb{N}\right\}$ *be a sequence nonexpansive self-mappings on H such that* $F(\mathcal{T}) \subseteq \cap_{n\in\mathbb{N}} F\left(S_{t_n}\right)$*. For* $x_0 \in H$*, let* $\{x_n\}$ *be a sequence in H generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) S_{t_n}(x_n) + e_n, \ n \geq 0 \tag{3.4}$$

where $\{\alpha_n\} \subset (0, 1]$, $\{\beta_n\} \subset [0, 1]$, *and* $\{e_n\}$ *is an error sequence in H satisfying the following conditions:*

(R1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(R2) $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Assume that $0 < \gamma < \bar{\gamma}/\alpha$ *and that* $\left\{S_{t_n}(x_n)\right\}$*is an approximating fixed point sequence of family* $\mathcal{T}$*. Assume that* $x^* \in F(\mathcal{T})$*, which solves the variational inequality (3.2). Then* $\{x_n\}$ *strongly converges to* x^**.*

Proof. Set $y_n = S_{t_n}(x_n)$. We divide the proof into three parts.

Step 1. Show the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

Noticing that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we may assume, with no loss of generality, that $\frac{\alpha_n}{1-\beta_n} < \|A\|^{-1}$ for all $n \geq 0$. From Lemma 2.7, we know that $||(1 - \beta_n) I - \alpha_n A|| \leq (1 - \beta_n - \alpha_n \bar{\gamma})$. Noticing that $x^* \in F(\mathcal{T})$, which solves the variational inequality (3.2). By assumption ($R2$), we have that $\{\frac{||e_n||}{\alpha_n}\}$ is bounded. Then, there exists a nonnegative real number K such that

$$\left\|\gamma f\left(x^*\right) - Ax^*\right\| + \frac{\|e_n\|}{\alpha_n} \leq K \,\text{for all}\, n \geq 0.$$

From (3.4), we have

$$\begin{aligned}
&||x_{n+1} - x^*|| \\
&= ||\alpha_n \left(\gamma f(x_n) - Ap\right) + \beta_n \left(x_n - x^*\right) + ((1 - \beta_n) I - \alpha_n A) \left(S_{t_n}(x_n) - x^*\right) + e_n|| \\
&\leq \alpha_n||\gamma f(x_n) - Ax^*|| + \beta_n||x_n - x^*|| + (1 - \beta_n - \alpha_n\bar{\gamma}) ||S_{t_n}(x_n) - x^*|| + ||e_n|| \\
&\leq \alpha_n\gamma||f(x_n) - f\left(x^*\right)|| + \alpha_n||\gamma f\left(p\right) - Ap|| + \beta_n||x_n - x^*|| \\
&+ (1 - \beta_n - \alpha_n\bar{\gamma}) ||x_n - x^*|| + ||e_n|| \\
&\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] ||x_n - x^*|| + \alpha_n \left(||\gamma f\left(p\right) - Ap|| + \frac{||e_n||}{\alpha_n}\right) \\
&\leq [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] ||x_n - x^*|| + \alpha_n K.
\end{aligned}$$

By simple inductions, we see that

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{K}{\bar{\gamma} - \gamma\alpha}\right\} =: R, \tag{3.5}$$

which yields that the sequence $\{x_n\}$ is bounded. Note that

$$\|y_n - x^*\| \leq \|x_n - x^*\|,$$

and hence the sequence $\{y_n\}$ is bounded.

Step 2. Show that

$$\limsup_{n\to\infty} \langle (\gamma f - A) x^*, y_n - x^* \rangle \leq 0,$$

where x^* is the solution of the variational inequality (3.2).

Let D be the ball centered in x^* and with radius R, i.e.,

$$D := \left\{w \in H : \|w - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{K}{\bar{\gamma} - \gamma\alpha}\right\}\right\}. \tag{3.6}$$

From (3.5) we see that D is a nonempty closed convex bounded subset of H which is $T(t)$-invariant for each $t \in [0, \infty)$ and contain $\{x_n\}$. Therefore, we assume, without loss of generality, $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ is a family nonexpansive self-mappings on D.

Taking a suitable subsequence $\{y_{n_i}\}$ of $\{y_n\}$, we see that

$$\limsup_{n\to\infty} \langle (\gamma f - A) x^*, y_n - x^* \rangle = \lim_{i\to\infty} \langle (\gamma f - A) x^*, y_{n_i} - x^* \rangle.$$

Since the sequence $\{y_n\}$ is also bounded, we may assume that $y_{n_i} \rightharpoonup \bar{x}$. Note that $\{y_n\}$ is an approximating fixed point sequence of family $\mathcal{T}$, i.e.,

$$\lim_{n\to\infty} \|y_n - T(h) y_n\| = 0 \text{ for all } 0 \leq h < \infty. \tag{3.7}$$

Using (3.7) we obtain, from the demiclosedness principle, that $\bar{x} \in F(\mathcal{T})$. Therefore, we have

$$\limsup_{n\to\infty} \langle (\gamma f - A) x^*, y_n - x^* \rangle = \langle (\gamma f - A) x^*, \bar{x} - x^* \rangle \leq 0. \tag{3.8}$$

On the other hand, we have

$$\|x_{n+1} - y_n\| \leq \alpha_n \|\gamma f(x_n) - Ax_n\| + \beta_n \|x_n - y_n\|.$$

From the assumption $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ that $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$, which combines with (3.8) gives that

$$\limsup_{n\to\infty} \langle (\gamma f - A) x^*, x_{n+1} - x^* \rangle \leq 0.$$

Step 3. Show $x_n \to x^*$ as $n \to \infty$.

Since the sequence $\{x_n\}$ is bounded, we may assume a nonnegative real number L such that $||x_n - x^*|| \leq L$ for all $n \geq 0$. Note that

$$
\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&= \langle \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(y_n - x^*) + e_n, x_{n+1} - x^* \rangle \\
&= \alpha_n \langle \gamma f(x_n) - Ax^*, x_{n+1} - x^* \rangle + \beta_n \langle x_n - x^*, x_{n+1} - x^* \rangle \\
&\quad + \langle ((1-\beta_n)I - \alpha_n A)(y_n - x^*) + e_n, x_{n+1} - x^* \rangle \\
&\le \alpha_n \left(\gamma \langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle + \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \right) \\
&\quad + \beta_n ||x_n - x^*|| \, ||x_{n+1} - x^*|| + ||(1-\beta_n)I - \alpha_n A|| \, ||y_n - x^*|| \, ||x_{n+1} - x^*|| + ||e_n|| L \\
&\le \alpha_n \alpha \gamma ||x_n - x^*|| \, ||x_{n+1} - x^*|| + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle \\
&\quad + \beta_n ||x_n - x^*|| \, ||x_{n+1} - x^*|| + (1 - \beta_n - \alpha_n \bar{\gamma}) ||x_n - x^*|| \, ||x_{n+1} - x^*|| + ||e_n|| L \\
&= [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] ||x_n - x^*|| \, ||x_{n+1} - x^*|| + \alpha_n \left\langle \gamma f(x^*) - Ax^*, \ x_{n+1} - x^* \right\rangle + ||e_n|| L \\
&\le \frac{1 - \alpha_n(\bar{\gamma} - \gamma\alpha)}{2} \left(||x_n - x^*||^2 + ||x_{n+1} - x^*||^2 \right) + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle + ||e_n|| L \\
&\le \frac{1 - \alpha_n(\bar{\gamma} - \gamma\alpha)}{2} ||x_n - x^*||^2 + \frac{1}{2} ||x_{n+1} - x^*||^2 + \alpha_n \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle + ||e_n|| L.
\end{aligned}
$$

It follows that

$$
||x_{n+1} - x^*||^2 \le [1 - \alpha_n(\bar{\gamma} - \gamma\alpha)] ||x_n - x^*||^2 + \alpha_n \left(2 \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^* \rangle + \frac{2||e_n||}{\alpha_n} L \right).
$$

By using Lemma 2.8, we can obtain the desired conclusion easily. □

4. Applications

4.1. Applications to zeros of maximal monotone operators

Let H be a real Hilbert space. Let $\mathbb{A} \subset H \times H$ be an operator on H. The set $D(\mathbb{A})$ defined by $D(\mathbb{A}) = \{x \in H : \mathbb{A}x \neq \emptyset\}$ is called the domain of $\mathbb{A}$, the set $R(\mathbb{A})$ defined by $R(\mathbb{A}) = \cup_{x \in X} \mathbb{A}x$ is called the range of $\mathbb{A}$ and the set $G(\mathbb{A})$ defined by $G(\mathbb{A}) = \{(x, y) \in H \times H : x \in D(\mathbb{A}), y \in \mathbb{A}x\}$ is called the graph of $\mathbb{A}$. An operator $\mathbb{A} \subset H \times H$ with domain $D(\mathbb{A})$ is said to be monotone if for each $x_i \in D(\mathbb{A})$ and $y_i \in \mathbb{A}x_i (i = 1, 2)$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \ge 0$. A monotone operator $\mathbb{A}$ is said to be maximal monotone if the graph $G(\mathbb{A})$ is not properly contained in the graph of any other monotone operator on H. If $\mathbb{A} : H \to 2^H$ is maximal monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^{\mathbb{A}} : H \to H$ by $J_\lambda^{\mathbb{A}} = (I + \lambda\mathbb{A})^{-1}$. It is called the resolvent of $\mathbb{A}$. Let $\mathcal{N}(\mathbb{A}) = \mathbb{A}^{-1}0 = \{x \in D(\mathbb{A}) : 0 \in \mathbb{A}x\}$. It is known that $\mathcal{N}(\mathbb{A})$ is closed and convex.

Lemma 4.1. ([22]) *Let $\mathbb{A} \subset H \times H$ be a maximal monotone operator. Then*

$$
\frac{1}{r} ||J_t x - J_r^{\mathbb{A}} J_t^{\mathbb{A}} x|| \le \frac{1}{t} ||x - J_t^{\mathbb{A}} x|| \text{ for all } x \in H \text{ and } r, \ t > 0.
$$

Proposition 4.2 shows that the family $\{J_t^{\mathbb{A}} : t > 0\}$ of resolvent operators of a maximal monotone operator $\mathbb{A}$ enjoys property $(\mathcal{A})$.

Proposition 4.2. *Let $\mathbb{A} \subset H \times H$ be a maximal monotone operator. Let $\{z_t\}_{t>0} \in \mathcal{B}(H)$ such that $\left\|z_t - J_t^{\mathbb{A}} z_t\right\| \to 0$ as $t \to \infty$. Then $\left\|z_t - J_r^{\mathbb{A}} z_t\right\| \to 0$ as $t \to \infty$ for each $r > 0$.*

Proof. Let r, $t > 0$. By Lemma 4.1, we have

$$
\frac{1}{r} ||J_t^{\mathbb{A}} z_t - J_r^{\mathbb{A}} J_t^{\mathbb{A}} z_t|| \le \frac{1}{t} ||z_t - J_t^{\mathbb{A}} z_t||. \tag{4.1}
$$

Using (4.1), we have

$$\begin{aligned}||z_t - J_r^{\mathbb{A}} z_t|| &\le ||z_t - J_t^{\mathbb{A}} z_t|| + ||J_t^{\mathbb{A}} z_t - J_r^{\mathbb{A}} J_t^{\mathbb{A}} z_t|| + ||J_r^{\mathbb{A}} J_t^{\mathbb{A}} z_t - J_r^{\mathbb{A}} z_t|| \\ &\le \left(2 + \frac{r}{t}\right) ||z_t - J_t^{\mathbb{A}} z_t|| \to 0 \text{ as } t \to \infty.\end{aligned}$$

□

Given a monotone operator $\mathbb{A} \subset H \times H$, we consider the following problem of finding $z \in H$ such that:

$0 \in \mathbb{A}z$. (P)

The Problem (P) can be regarded as a unified formulation of several important problems. For an appropriate choice of the operator A, Problem (P) covers a wide range of mathematical applications; for example, variational inequalities, complementarity problems, and non-smooth convex optimization. Problem (P) has applications in physics, economics, and in several areas of engineering. Therefore, one of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of maximal monotone operators. One method for finding zeros of maximal monotone operators is the *proximal point algorithm*. Let A be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = J_{r_n}^{\mathbb{A}} x_n \text{ for all } n \in \mathbb{N}, \tag{4.2}$$

where $\{r_n\}$ is a regularization sequence in $(0, \infty)$. Note that (4.2) is equivalent to

$$0 \in \frac{1}{r_n}(x_{n+1} - x_n) + \mathbb{A}x_{n+1} \text{ for all } n \in \mathbb{N}.$$

This was first introduced by Martinet [23]. If $\psi : H \to \mathbb{R} \cup \{\infty\}$ is a proper lower semicontinuous convex function, then the algorithm reduces to

$$x_{n+1} = \arg\min_{y \in H} \left\{\psi(y) + \frac{1}{2r_n}||x_n - y||^2\right\} \text{ for all } n \in \mathbb{N}.$$

Rockafellar [24] studied the proximal point algorithm in the framework of Hilbert space and he proved the following:

Theorem 4.3. *Let H be a Hilbert space and $\mathbb{A} \subset H \times H$a maximal monotone operator. Let $\{x_n\}$ be a sequence in H defined by (4.2), where $\{r_n\}$ is a sequence in* $(0, \infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$. *If $\mathbb{A}^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of* $\mathbb{A}^{-1}0$.

Rockafellar [24] has given a more practical method which is an inexact variant of the method

$$e_n \in x_n - x_{n-1} + r_n A x_n,$$

where $\{e_n\}$ is regarded as an error sequence. The method is called inexact proximal point algorithm. It was shown in Rockafellar [24] that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty} ||e_n|| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in A(z)$. In 2002, Xu [12] modified the proximal point algorithm for solving Problem (P) and gave strong convergence of the algorithm in a Hilbert space setting under the same assumption $\sum_{n=1}^{\infty} ||e_n|| < \infty$.

The criteria $\sum_{n=1}^{\infty} ||e_n|| < \infty$ imposed for convergence of inexact proximal point algorithms (see [12,24]) is somewhat undesirable, because it impose increasing precision along the iterative process. This brings us to the following natural question:

Question 4.4. *Is it possible to further modify inexact proximal point algorithm without the assumption* $\sum_{n=1}^{\infty} ||e_n|| < \infty$*, so that it can generate a strongly convergent sequence?*

Recently, Sahu and Yao [25] introduced and studied the prox-Tikhonov method for solving Problem (P) in the Banach space setting and they partially answered Question 4.4. We now establish more general results in the Hilbert space setting:

Theorem 4.5. *Let H be a real Hilbert space H. Let $\mathbb{A} \subset H \times H$ be a maximal monotone operator with $\mathcal{N}(\mathbb{A}) \neq \emptyset$, $f : H \to H$ an α-contraction and A a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and that $\{b_t : t > 0\}$ is a net in $(0, ||A||^{-1})$ such that $\lim_{t\to\infty} b_t = 0$. Then $\{x_t\}$ defined by*

$$x_t = b_t \gamma f(x_t) + (I - b_t A) J_t^{\mathbb{A}} x_t.$$

strongly converges as $t \to \infty$ to $x^* \in \mathcal{N}(\mathbb{A})$, where $x^* \in P_{\mathcal{N}(\mathbb{A})}(I - A + \gamma f)x^*$ is a solution of the following variational inequality:

$$\langle (\gamma f - A)x^*, p - x^* \rangle \leq 0, \ \forall p \in \mathcal{N}(\mathbb{A}). \tag{4.3}$$

Proof. Set $T(t) := J_t^{\mathbb{A}}$ for $t > 0$. Then $\{T(t) : t > 0\}$ is a family of nonexpansive mappings with $F(T(t)) = \mathcal{N}(\mathbb{A})$ for each $t > 0$. Proposition 4.2 shows that the family $\{J_t^{\mathbb{A}} : t > 0\}$ of resolvent operators enjoys property $(\mathcal{A})$. Therefore, Theorem 4.5 follows from Theorem 3.2. □

Theorem 4.6. *Let H be a real Hilbert space H. Let $\mathbb{A} \subset H \times H$ be a maximal monotone operator with $\mathcal{N}(\mathbb{A}) \neq \emptyset$, $f : H \to H$ an α-contraction and A a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\{t_n\}$ is a positive real divergent sequence. For $x_0 \in H$, let $\{x_n\}$ be a sequence in H generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) J_{t_n}^{\mathbb{A}}(x_n) + e_n, \quad n \geq 0$$

where $\{\alpha_n\} \subset (0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{e_n\}$ is an error sequence in H satisfying conditions $(R1)$ and $(R2)$. Then $\{x_n\}$ strongly converges to x^*, where $x^* = P_{\mathcal{N}(\mathbb{A})}(I - A + \gamma f)x^*$ is a solution of the variational inequality (4.3).

Proof. Set $S_{t_n} := J_{t_n}^{\mathbb{A}}$ and $y_n := S_{s_n}(x_n)$. Then it remains to show that $\{y_n\}$ is an approximating fixed point sequence of the family $\{J_t^{\mathbb{A}} : t > 0\}$ of resolvent operators of $\mathbb{A}$. As in the proof of Theorem 3.3, one can show that $\{x_n\}$ and $\{y_n\}$ are bounded. Then, there positive real number M such that $\|x_n - J_{t_n}^{\mathbb{A}} x_n\| \leq M$ for all $n \geq 0$. For any fixed $r > 0$, by Lemma 4.1, we have

$$\begin{aligned} ||J_{t_n}^{\mathbb{A}} x_n - J_r^{\mathbb{A}} J_{t_n}^{\mathbb{A}} x_n|| &\leq \frac{r}{t_n} ||x_n - J_{t_n}^{\mathbb{A}} x_n|| \\ &\leq \frac{r}{t_n} M. \end{aligned}$$

Thus, in particular, we derive

$$||y_n - J_r^{\mathbb{A}} y_n|| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

for all $r > 0$. Therefore, Theorem 4.6 follows from Theorem 3.3. □

Theorem 4.6 is more general than results of Kamimura and Takahashi [26] and Xu [12]. In particular, Theorem 4.6 provides an affirmative answer of Question 4.4 in the context of finding solution of variational inequality (4.3).

4.2. Applications to equilibrium problems

Let H be a Hilbert space and $G : H \times H \to \mathbb{R}$ be an equilibrium function, that is

$$G(u, u) = 0 \qquad \text{for every } u \in H.$$

The equilibrium problem is defined as follows,

$$\text{Find } \tilde{x} \in H \text{ such that } G(x, y) \geq 0 \text{ for all } y \in H.$$

A solution $\tilde{x}$ of the equilibrium problem is called an equilibrium point and the set of all equilibrium points will be denoted by $EP(G)$. The topic has been considered by several authors (see [27,28]). We shall assume some mild conditions over G in such a way that results can be applied in several cases including optimization problems, fixed point problems, variational problems, variational inequality problems, and convex vector minimization problems [29,30].

Lemma 4.7. ([29]) *Let C be a nonempty closed convex subset of H and $G : C \times C \to \mathbb{R}$ satisfy,*

(A1) *for all $x \in C$, $G(x, x) = 0$;*

(A2) *G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;*

(A3) *for all $x, y, z \in C$,*

$$\limsup G(tz + (1 - t)x, y) \leq G(x, y) \text{ as } t \to 0;$$

(A4) *for all $x \in C$, $y \alpha G(x, y)$ is convex and lower semicontinuous.*

For $\times \in C$ and $r > 0$, set $S_r : H \to C$ to be the resolvent for G,

$$S_r(x) := \left\{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\},$$

then S_r is well defined and the following hold:

(1) *S_r is single-valued;*

(2) *S_r is firmly nonexpansive, i.e.,*

$$||S_r x - S_r y||^2 \leq \langle S_r x - S_r y, x - y \rangle,$$

for all $x, y \in H$;

(3) *$F(S_r) = EP(G)$;*

(4) *$EP(G)$ is closed and convex.*

In order to show that the family $\{S_t : t > 0\}$ of resolvent operators of G enjoys property $(\mathcal{A})$, we need the following technical lemma.

Lemma 4.8. *Let G be an equilibrium function satisfying the assumptions of Lemma 4.7. Then*

$$||S_t(x) - S_rS_t(x)|| \le \frac{r}{t}||x - S_t(x)||$$

for all $x \in H$ *and* $r, t > 0$.

Proof. Let $x, y \in H$ and $r, t > 0$. By the definition of S_t, we have

$$G(S_t(x),\ z) + \frac{1}{t}\langle z - S_t(x),\ S_t(x) - x\rangle \ \ge\ 0, \forall z \in H \tag{4.4}$$

and

$$G(S_r(y),\ z) + \frac{1}{r}\langle z - S_r(y),\ S_r(y) - y\rangle \ge 0, \forall z \in H. \tag{4.5}$$

Put $z = S_r(y)$ in (4.4) and $z = S_t(x)$ in (4.5), we obtain

$$G(S_t(x),\ S_r(y)) + \frac{1}{t}\langle S_r(y) - S_t(x), S_t(x) - x\rangle \ge 0 \tag{4.6}$$

and

$$G(S_r(y), S_t(x)) + \frac{1}{r}\langle S_t(x) - S_r(y), S_r(y) - y\rangle \ge 0, \tag{4.7}$$

respectively. Since G is monotone, from (4.6) and (4.7), we have

$$\left\langle S_t(x) - S_r(y), \frac{S_r(y) - y}{r} - \frac{S_t(x) - x}{t}\right\rangle \ge 0. \tag{4.8}$$

Set $y = S_t(x)$ in (4.8), we get

$$\left\langle S_t(x) - S_rS_t(x),\ \frac{S_rS_t(x) - S_t(x)}{r} - \frac{S_t(x) - x}{t}\right\rangle \ge 0$$

and hence

$$\frac{1}{r}||S_t(x) - S_rS_t(x)||^2 \le \left\langle S_t(x) - S_rS_t(x), \frac{x - S_t(x)}{t}\right\rangle \le \frac{1}{t}||S_t(x) - S_rS_t(x)||\,||x - S_t(x)||.$$

Therefore,

$$||S_t(x) - S_rS_t(x)|| \ \le \frac{r}{t}||x - S_t(x)||.$$

□

From this, we deduce the property $(\mathcal{A})$ for the family $\{S_t : t > 0\}$ of resolvent operators of G.

Lemma 4.9. *Let G be an equilibrium function satisfying the assumptions of Lemma 4.7. Then the family* $\{S_t : t > 0\}$ *enjoys property* $(\mathcal{A})$.

Proof. Let $\{z_t\} \in \mathcal{B}(H)$ such that $z_t - S_tz_t \to 0$. Then, for any fixed $r > 0$,

$$\begin{aligned} ||z_t - S_rz_t|| &\le\ ||z_t - S_tz_t|| + ||S_tz_t - S_rS_tz_t|| + ||S_rS_tz_t - S_rz_t|| \\ &\le \left(2 + \frac{r}{t}\right)||z_t - S_tz_t|| \end{aligned}$$

by nonexpansivity and Lemma 4.8. In particular, we derive that $z_t - S_rz_t \to 0$ as $t \to \infty$. □

From this last and from Theorem 3.2, we have

Theorem 4.10. *Let H be a real Hilbert space H. Let $G : H \times H \to \mathbb{R}$be an equilibrium function satisfying the assumptions of Lemma 4.7 and let $\{S_t : t > 0\}$ be the family of resolvent operators for G. Let $f : H \to H$ be an α-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $EP(G) \neq \emptyset$ and $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{b_t : t > 0\}$ be a net in $(0, ||A||^{-1})$ such that $\lim_{t\to\infty} b_t = 0$. Then $\{x_t\}$ defined by*

$$x_t = b_t \gamma f(x_t) + (I - b_t A) S_t x_t. \tag{4.9}$$

strongly converges as $t \to \infty$ to $x^* \in EP(G)$, where $x^* = P_{EP(G))}(I - A + \gamma f)x^*$ is a solution of the following variational inequality:

$$\langle (\gamma f - A)x^*, p - x^* \rangle \leq 0, \quad \forall p \in EP(G). \tag{4.10}$$

Proof. Note that $\{S_t : t > 0\}$ is a family of resolvent operators for G such that $F(S_t) = EP(G)$ for each $t > 0$. Lemma 4.9 shows that the family $\{S_t : t > 0\}$ of resolvent operators of G enjoys property $(\mathcal{A})$. Therefore, Theorem 4.10 follows from Theorem 3.2. □

Theorem 4.11. *Let H be a real Hilbert space H. Let $G : H \times H \to \mathbb{R}$be an equilibrium function satisfying the assumptions of Lemma 4.7 and let $\{S_t : t > 0\}$ be the family of resolvent operators for G such that $EP(G) \neq \emptyset$. Let $f : H \to H$ be an α-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\{t_n\}$ is a positive real divergent sequence. For $x_0 \in H$, let $\{x_n\}$ be a sequence in H generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) S_{t_n}(x_n) + e_n, \quad n \geq 0 \tag{4.11}$$

where $\{\alpha_n\} \subset (0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{e_n\}$ is an error sequence in H satisfying conditions $(R1)$ and $(R2)$. Then $\{x_n\}$ strongly converges to $x^* \in EP(G)$, where $x^* = P_{EP(G))}(I - A + \gamma f)x^*$ is a solution of the variational inequality (4.10).

Proof. Set $y_n := S_{t_n}(x_n)$. Then it remains to show that $\{y_n\}$ is an approximating fixed point sequence of the family $\{S_t : t > 0\}$ of resolvent operators of G. As in the proof of Theorem 3.3, one can show that $\{z_n\}$ and $\{S_{t_n}(x_n)\}$ are bounded. Then, there positive real number M such that $||x_n - S_{t_n}x_n|| \leq M$ for all $n \geq 0$. For any fixed $r > 0$, by Lemma 4.8, we have

$$||S_{t_n}x_n - S_r S_{t_n} x_n|| \leq \frac{r}{t_n} M.$$

In particular, we derive $||y_n - S_r y_n|| \to 0$ as $n \to \infty$. for all $r > 0$. Therefore, Theorem 4.11 follows from Theorem 3.3. □

Theorem 4.11 extends the corresponding result of Song et al. [31] in the context of the variational inequality (4.10).

4.3. Applications to common fixed point problems

In this section, we deduce some results by using Theorems 3.2 and 3.3. As a direct consequence of Theorem 4.12, we first have the following result.

Theorem 4.12. *Let H be a real Hilbert space H and $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$be a nonexpansive semigroup on H such that $F(\mathcal{T}) \neq \emptyset$. Let $f : H \to H$ be an α-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and that $\{b_t : t > 0\}$ is a net in $(0, ||A||^{-1})$ such that $\lim_{t\to\infty} b_t = 0$. Then $\{x_t\}$ define by*

$$x_t = b_t\gamma f(x_t) + (I - b_t A)\frac{1}{t}\int_0^t T(s)\, x_t ds.$$

strongly converges as $t \to \infty$ *to* $x^* \in F(\mathcal{T})$, *where* $x^* = P_{F(\mathcal{T})}(I - A + \gamma f)x^*$*is a solution of the variational inequality (3.2).*

Proof. Example 2.12 implies that nonexpansive semigroup$\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ has property $(\mathcal{A})$ with respect to a nonexpansive semigroup $\{\sigma_t : t \in \mathbb{R}^+\}$. Therefore, Theorem 4.12 follows from Theorem 3.2. □

Remark 4.13. *Theorem 4.12 which include the corresponding results of Shioji and Takahashi* [16]*as a special case is reduced to Plubtieng and Punpaeng* [14]*when* $A = I$, *the identity mapping and* $\gamma = 1$.

Theorem 4.14. *Let H be a real Hilbert space H and* $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$*a nonexpansive semigroup such that* $F(\mathcal{T}) \ne \emptyset$. *Let* $f : H \to H$ *be an* α*-contraction and A a strongly positive linear bounded self-adjoint operator on H with the coefficient* $\bar{\gamma} > 0$. *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$ *and* $\{t_n\}$ *is a positive real divergent sequence. For* $x_0 \in H$, *let* $\{x_n\}$ *be a sequence in H generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + e_n, \quad n \ge 0$$

where $\{\alpha_n\} \subset (0, 1]$, $\{\beta_n\} \subset [0, 1]$ *and* $\{e_n\}$ *is an error sequence in H satisfying conditions* $(R1)$ *and* $(R2)$. *Then* $\{x_n\}$ *strongly converges to* $x^* \in F(\mathcal{T})$, *where* $x^* = P_{F(\mathcal{T})}(I - A + \gamma f)x^*$*is a solution of the variational inequality (3.2).*

Proof. For each$n \in \mathbb{N}$, define$y_n = S_{t_n}(x_n)$. Note that $\{y_n\}$ is in a bounded set D defined by (3.6). As in the the proof of Theorem 3.3, $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$ is a semigroup of nonexpansive self-mappings on D. It follows from Lemma 2.1 that $\{y_n\}$ is an approximating fixed point sequence of semigroup $\mathcal{T}$. □

Remark 4.15. *If* $\gamma = 1$ *and* $A = I$, *the identity mapping, then Corollary 2.4 is reduced to Theorem 3.3 of Plubtieng and Punpaeng* [14].

If the sequence $\{\beta_n\} \equiv 0$, then Theorem 4.14 reduces to the following:

Corollary 4.16. *Let H be a real Hilbert space H and* $\mathcal{T} = \{T(t) : 0 \le t < \infty\}$*be a nonexpansive semigroup such that* $F(\mathcal{T}) \ne \emptyset$. *Let* $f : H \to H$ *be an* α*-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient* $\bar{\gamma} > 0$. *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$ *and* $\{t_n\}$ *is a positive real divergent sequence. For* $x_0 \in H$, *let* $\{x_n\}$ *be a sequence in H generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + e_n, \quad n \ge 0$$

where $\{\alpha_n\} \subset (0, 1]$ *and* $\{e_n\}$ *is an error sequence in H satisfying the following conditions:*

(R3) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(R4). $\lim\limits_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$

Then $\{x_n\}$ *strongly converges to* $x^* \in F(\mathcal{T})$, *where* $x^* = P_{F(\mathcal{T})}(I - A + \gamma f)x^*$*is a solution of the variational inequality (3.2).*

Remark 4.17. *Corollary 4.16 includes Theorem 2 of Shimizu and Takahashi* [17]*as a special case.*

Remark 4.18. *Theorem 2.2 and Corollary 4.16 improve Theorems 3.2 and 3.4 of Marino and Xu* [10]*from a single nonexpansive mapping to a nonexpansive semigroup, respectively.*

Using [[17], Lemma 1], we derive the following result, which generalizes Theorem 1 of Shimizu and Takahashi [17].

Corollary 4.19. *Let H be a real Hilbert space H and let S,* $T : H \to H$ *be two commuting nonexpansive mappings such that* $F(S) \cap F(T) \neq \emptyset$. *Let* $f : H \to H$ *be an* α*-contraction and A be a strongly positive linear bounded self-adjoint operator on H with the coefficient* $\bar{\gamma} > 0$ *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$. *For* $x_0 \in H$, *let* $\{x_n\}$ *be a sequence in H generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} S^i T^j x_n + e_n, \quad n \geq 0$$

where $\{\alpha_n\} \subset (0, 1]$ *and* $\{e_n\}$ *is an error sequence in H satisfying conditions* (*R*3) *and* (*R*4). *Then* $\{x_n\}$ *strongly converges to* $x^* \in F(S) \cap F(T)$, *where* $x^* = P_{F(S)\cap F(T)}(I - A + \gamma f)x^*$ *is a solution of the following variational inequality:*

$$\langle (\gamma f - A)x^*, \ p - x^* \rangle \leq 0, \quad \forall p \in F(S) \cap F(T).$$

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References
1. Browder, FE: Nonexpansive nonlinear operators in a Banach space. Proc Nat Acad Sci USA. **54**, 1041–1044 (1965). doi:10.1073/pnas.54.4.1041
2. Browder, FE: Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch Rat Mech Anal. **24**, 82–90 (1967)
3. Reich, S: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J Math Anal Appl. **75**, 287–292 (1980). doi:10.1016/0022-247X(80)90323-6
4. Halpern, B: Fixed points of nonexpansive maps. Bull Am Math Soc. **73**, 957–961 (1967). doi:10.1090/S0002-9904-1967-11864-0
5. Lions, P-L: Approximation de points fixes de contractions. CR Acad Sci Paris Ser A-B. **284**, 1357–1359 (1977)
6. Wittmann, R: Approximation of fixed points of nonexpansive mappings. Arch Math. **58**, 486–491 (1992). doi:10.1007/BF01190119
7. Deutsch, F, Yamada, I: Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings. Numer Funct Anal Optim. **19**, 33–56 (1998)
8. Li, S, Li, L, Su, Y: General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space. Nonlinear Anal. **70**, 3065–3071 (2009). doi:10.1016/j.na.2008.04.007
9. Marino, G, Colao, V, Qin, X, Kang, SM: Strong convergence of the modified Mann iterative method for strict pseudo-contractions. Comput Math Appl. **57**, 455–465 (2009). doi:10.1016/j.camwa.2008.10.073
10. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J Math Anal Appl. **318**, 43–52 (2006). doi:10.1016/j.jmaa.2005.05.028
11. Xu, HK: An iterative approach to quadratic optimization. J Optim Theory Appl. **116**, 659–678 (2003). doi:10.1023/A:1023073621589
12. Xu, HK: Iterative algorithms for nonlinear operators. J Lond Math Soc. **66**, 240–256 (2002). doi:10.1112/S0024610702003332

13. Yamada, I, Ogura, N, Yamashita, Y, Sakaniwa, K: Quadratic approximation of fixed points of nonexpansive mappings in Hilbert spaces. Numer Funct Anal Optim. **19**, 165–190 (1998). doi:10.1080/01630569808816822
14. Plubtieng, S, Punpaeng, R: Fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces. Math Comput Model. **48**, 279–286 (2008). doi:10.1016/j.mcm.2007.10.002
15. Cianciaruso, F, Marino, G, Muglia, L: Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces. J Optim Theory Appl. **146**(2), 491–509 (2010). doi:10.1007/s10957-009-9628-y
16. Shioji, N, Takahashi, W: Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces. Nonlinear Anal. **34**, 87–99 (1998). doi:10.1016/S0362-546X(97)00682-2
17. Shimizu, T, Takahashi, W: Strong convergence to common fixed points of families of nonexpansive mappings. J Math Anal Appl. **211**, 71–83 (1997). doi:10.1006/jmaa.1997.5398
18. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed point theory for Lipschitzian-type mappings with applications, Series. In Topological Fixed Point Theory and Its Applications, vol. 6,Springer, New York (2009)
19. Geobel, K, Kirk, WA: Topics in metric fixed point theory. In Cambridge Stud Adv Math, vol. 28,Cambridge Univ. Press, Cambridge (1990)
20. Aleyner, A, Reich, S: An explicit construction of sunny nonexpansive retractions in Banach spaces. Fixed Point Theory and Applications. **3**, 295–305 (2005)
21. Edelstein, M, O'Brien, RC: Nonexpansive mappings, asymptotic regularity and successive approximations. J Lond Math Soc. **3**, 547–554 (1978)
22. Takahashi, W: Nonlinear functional analysis, Fixed point theory and its applications. Yokohama Publishers, Yokohama (2000)
23. Martinet, B: Regularisation dinequations variationnelles par approximations successives. Rev FranMcaise Inf Recherche Operationnelle. **4**, 154–158 (1970)
24. Rockafellar, RT: Monotone operators and the proximal point algorithm. SIAM J Control Optim. **14**, 877–898 (1976). doi:10.1137/0314056
25. Sahu, DR, Yao, JC: The prox-Tikhonov regularization method for the proximal point algorithm in Banach spaces. J Glob Optim. **51**, 641–655 (2011). doi:10.1007/s10898-011-9647-8
26. Kamimura, S, Takahashi, W: Approximating solutions of maximal monotone operators in Hilbert space. J Approx Theory. **106**, 226–240 (2000). doi:10.1006/jath.2000.3493
27. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math Stud. **63**, 123–145 (1994)
28. Oettli, W: A remark on vector-valued equilibria and generalized monotonicity. Acta Math Vietnam. **22**, 215–221 (1997)
29. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J Nonlinear Convex Anal. **6**(1), 117–136 (2005)
30. Iusem, AN, Sosa, W: Iterative algorithms for equilibrium problems. Optimization. **52**(3), 301–316 (2003). doi:10.1080/0233193031000120039
31. Song, Y, Kang, JI, Cho, YJ: On iterations methods for zeros of accretive operators in Banach spaces. Appl Math Comput. **216**, 1007–1017 (2010). doi:10.1016/j.amc.2010.01.124

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Strong convergence theorems for approximating common fixed points of families of nonexpansive mappings and applications

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Abstract An implicit algorithm for finding common fixed points of an uncountable family of nonexpansive mappings is proposed. A new inexact iteration method is also proposed for countable family of nonexpansive mappings. Several strong convergence theorems based on our main results are established in the setting of Banach spaces. Both algorithms are applied for finding zeros of accretive operators and for solving convex minimization, split feasibility and equilibrium problems.

Keywords Fixed point · Nonexpansive mappings · Accretive operators · Resolvent operator split feasibility problem · Equilibrium problem

1 Introduction

From a theoretical point of view, many problems which arise from real-world applications can be translated into equivalent fixed point problems. This approach had been successfully applied in different topics, including convex minimization, split feasibility and equilibrium problems, as well as for finding zeros of accretive operators.

In order to find an approximate solution of the above mentioned problems a theoretical framework was developed in which the study of iterative methods in abstract spaces plays an important role. We also note that a qualitative analysis of the results obtained in the abstract model had only been studied in few papers (see, e.g., [23,38] and the references

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therein). However, a qualitative examination can be generally and more easily performed on the algorithm, when it is implemented for specific problems.

The aim of this paper is to study iterative algorithms for countable families of nonexpansive mappings in the setting of Hilbert and Banach spaces. We study implicit and explicit methods, and we prove the strong convergence of the generated sequences to a common fixed point of the family of mappings. To this end, for a family $\mathcal{T}$ of nonexpansive mappings, we introduce an asymptotic regularity condition, namely property $(\mathscr{A})$, which is weaker than the uniform asymptotic regularity introduced in [27]. We prove that important and frequently used families of mappings satisfy property $(\mathscr{A})$, so that our results apply. In particular, if $A \subset H \times H$ is a maximal monotone operator on a Hilbert space H and if $\{c_n\}_{n\in\mathbb{N}}$ is a sequence in $(0, \infty)$, the family $\{J^A_{c_n}\}_{n\in\mathbb{N}}$ of resolvents of A satisfies the property $(\mathscr{A})$. We mention that such family plays a fundamental role in convex analysis and it is a common tool to approximate zeros of the operator A.

One of the first and mostly celebrated result in this direction is due to Rockafellar [48]. In his work, he introduced and studied an inexact proximal point method which can be described as follows

$$\text{Take an arbitrary } z_0 \in H \quad \text{and} \quad z_{k+1} \approx J^A_{c_k} z_k.$$

Then the sequence $\{z_k\}$ weakly converges to a zero of A, provided that $\|z_{k+1} - J^A_{c_k} z_k\| \le \epsilon_k$ and that $\{\epsilon_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ satisfies

$$\sum_{k=1}^{\infty} \epsilon_k < \infty. \tag{1.1}$$

Later, Eckstein and Berstekas [25] extended a related result due to Gol'shtein and Tret'yakov [30] by introducing and studying the iterations

$$z_{k+1} = (1-\rho_k) z_k + \rho_k w_k,$$

where $\{\rho_k\}_{k\in\mathbb{N}} \subset (0, 2)$ and $\|w_k - J^A_{c_k} z_k\| \le \epsilon_k$. Moreover, they proved that under the assumption (1.1), the method weakly converges to a zero of A.

A question naturally arising from the above mentioned results is if it is possible to construct an iterative sequence which converges under weaker assumptions on the error sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$. Indeed, condition (1.1) is somewhat undesirable as it implies a fast increasing precision along the iterative process. In this paper, we formulate an inexact iterative algorithm for which the conditions on the error sequence can be relaxed. Moreover, our results hold in the more general setting of Banach spaces and accretive operators.

Let H be a Hilbert space and let $G : H \times H \to \mathbb{R}$ be a bifunction. The equilibrium problem for G is stated as

$$\text{find } \tilde{x} \in H \text{ such that } G(\tilde{x}, y) \ge 0 \quad \text{for all} \quad y \in H. \tag{1.2}$$

The above mentioned problem had been considered by several authors (see, for instance, [2,8–10,28,46]) and it contains as special cases optimization problems, problems of Nash equilibria and complementarity problems, among others.

In [24], it had been proved that under mild assumptions on G and for any $r > 0$, the mapping $S_r : H \to H$ defined by

$$x \mapsto \left\{ z \in H : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in H \right\}$$

is single-valued and nonexpansive. Moreover, the fixed point set $F(S_r)$ coincides with the solutions set of the problem (1.2). The family $\{S_r\}_{r\in(0,\infty)}$ is called the family of resolvents of G.

In this paper, we prove that the family $\{S_r\}_{r\in(0,\infty)}$ satisfies the property $(\mathscr{A})$, so our results apply in this setting. It is also shown that our algorithm can be successfully applied and with the same results for finding solutions to convex minimization and split feasibility problems.

For an arbitrary sequence of nonexpansive mappings $\{T_n\}_{n\in\mathbb{N}}$ which lacks of asymptotic regularity, a partial workaround had been introduced in [3] by constructing the sequence of nonexpansive auxiliary mappings $\{S_n\}_{n\in\mathbb{N}}$, defined by $S_n(x) := \sum_{k=1}^{n} \beta_{k,n} T_k(x)$, where $\{\beta_{k,n}\} \subset \mathbb{R}^+$ satisfies particular conditions. In the same paper, the improved asymptotic behaviour of the family $\{S_n\}$ is then used to prove a convergence result for an iterative sequence in Banach spaces. We suspect, although we have no proof, that the sequence $\{T_n\}_{n\in\mathbb{N}}$ satisfies the property $(\mathscr{A})$ with respect to the family $\{S_n\}_{n\in\mathbb{N}}$.

2 Preliminaries

Let C be a nonempty subset of a normed space X and let $T : C \to C$ be a nonexpansive mapping, i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The set C is called a retract of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a retraction of X onto C. A retraction P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each x in X and all $t \ge 0$. If a sunny retraction P is also nonexpansive, then C is said to be a sunny nonexpansive retract of X.

Remark 2.1 (cf. [50]) If X is a smooth Banach space, then there is at most one sunny nonexpansive retraction Q from X onto C.

Let C be a nonempty subset of a Banach space X, let $\mathscr{B}(C)$ denote the collection of all bounded subsets of C and let $S_1, S_2 : C \to X$ be two mappings. The deviation between S_1 and S_2 on $B \in \mathscr{B}(C)$ (see [51]), denoted by $\mathscr{D}_B(S_1, S_2)$, is defined by

$$\mathscr{D}_B(S_1, S_2) = \sup_{x\in B} \|S_1x - S_2x\|.$$

Throughout this paper, G denotes an unbounded subset of $\mathbb{R}^+ := [0, \infty)$ such that $s + t \in G$ for all $s, t \in G$ (often $G = \mathbb{N}$ or $\mathbb{R}^+$). Let $\mathcal{T} := \{T_s : s \in G\}$ be a family of mappings from C into itself. We denote by $F(\mathcal{T})$ the common fixed points set of $\mathcal{T}$, that is, $F(\mathcal{T}) = \bigcap_{t\in G} F(T_t)$. A family $\mathcal{T}$ is said to be *uniformly asymptotically regular on C* (for short u.a.r. on C) (see [1,6,27]) if

$$\lim_{t\in G, t\to\infty} (\sup_{x\in\tilde{C}} \|T_t x - T_s T_t x\|) = 0 \quad \text{for all} \quad s \in G \quad \text{and} \quad \tilde{C} \in \mathscr{B}(C).$$

A family $\mathcal{T} := \{T_s : s \in G\}$ satisfies the property $(\mathscr{A})$ if the following holds, for any $\{x_s\}_{s\in G} \in \mathscr{B}(C)$ which satisfies $x_s - T_s x_s \to 0$ as $s \to \infty$ we have that $x_s - T_t x_s \to 0$ as $s \to \infty$ for all $t \in G$.

Remark 2.2 If $\mathcal{T}$ is a singleton, i.e., $\mathcal{T} = \{T\}$, or $T_s = T$ for all s in G, then $\{T\}$ always satisfies the property $(\mathscr{A})$.

We further remark that the notion of uniform asymptotic regularity introduced by Edelstein and O'Brine [27] plays an important role for studying the property $(\mathscr{A})$ of nonlinear Lipschitzian-type operators. Indeed, if $\mathcal{T} := \{T_s : s \in G\}$ is a nonexpansive semigroup and a.u.r., then $\mathcal{T}$ has property $(\mathscr{A})$. Indeed, for any $\{y_s\}_{s\in G} \in \mathscr{B}(C)$ with $x_s - T_s x_s \to 0$ as $s \to \infty$, we have for all $t \in G$

$$\begin{aligned}\|y_s - T_t y_s\| &\le \|y_s - T_s y_s\| + \|T_s y_s - T_t T_s y_s\| + \|T_t T_s y_s - T_t y_s\| \\ &\le 2\|y_s - T_s y_s\| + \sup_{y\in\{y_\gamma:\gamma\in G\}} \|T_s y - T_t T_s y\| \to 0 \text{ as } s \to \infty.\end{aligned}$$

Let $\mathcal{G} = \{G_s : s \in G\}$ be a family of mappings from C into itself. We say the family $\mathcal{T} := \{T_t : t \in \mathbb{R}^+\}$ has property $(\mathscr{A})$ with respect to the family $\mathcal{G} = \{G_s : s \in G\}$ if the following holds:
for any $\{x_s\}_{s\in G} \in \mathscr{B}(C)$ which satisfies $x_s - G_s x_s \to 0$ as $s \to \infty$, we have $x_s - T_t x_s \to 0$ as $s \to \infty$ for all $t > 0$.

Remark 2.3 (1) If a family $\mathcal{T} := \{T_t : t \in \mathbb{R}^+\}$ has the property $(\mathscr{A})$, then $\mathcal{T}$ has the property $(\mathscr{A})$ with respect to itself.

(2) If a family $\mathcal{T} := \{T_t : t \in \mathbb{R}^+\}$ has the property $(\mathscr{A})$ with respect to a family $\mathcal{S} := \{T_n : n \in \mathbb{N}\}$, then the following holds:
for any $\{x_n\}_{n\in\mathbb{N}} \in \mathscr{B}(C)$ which satisfies $x_n - S_n x_n \to 0$ as $n \to \infty$, we have $x_n - T_t x_n \to 0$ as $n \to \infty$ for all $t > 0$.

We now give an example.

Example 2.4 Let C be a nonempty, closed and convex subset of the Banach space X and T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$. For each $t \in G$, and $b_t \in \mathbb{R}$ with $0 < a \le b_t \le b < 1$, define $G_t : C \to C$ by

$$G_t x = (1 - b_t)x + b_t T x \quad \text{for all} \quad x \in C.$$

Then T has property $(\mathscr{A})$ with respect to the family $\{G_t : t \in G\}$.

Proof Let $\{x_t\}_{t\in G} \in \mathscr{B}(C)$ such that $\|x_t - G_t(x_t)\| \to 0$ as $t \to \infty$. Note that

$$\|x_t - T x_t\| = b_t \|x_t - G_t(x_t)\|$$

and $0 < a \le b_t \le b < 1$ for all $t \in G$. Therefore, $\|x_t - T x_t\| \to 0$ as $t \to \infty$. □

Let l^∞ be the space of all bounded and real-valued sequences and let $(a_0, a_1, \ldots) \in l^\infty$. For a functional ϕ on l^∞, we write $\phi_n a_n$ instead of $\phi(a_0, a_1, \ldots)$.

Definition 2.5 A Banach limit LIM is a continuous and linear functional on l^∞ such that

(1) $\|LIM\| = LIM_n 1 = 1$,
(2) $LIM_n a_{n+1} = LIM_n a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

The normalized duality mapping J from X into 2^{X^*} is defined by

$$J(x) := \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\}.$$

Lemma 2.6 *Let X be a Banach space. Then for each $x, y \in X$, there exists $j(x + y) \in J(x + y)$ such that*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 2.7 (cf. [32, Lemma 1]) *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty, closed and convex subset of X and $\{x_n\}$ be a bounded sequence in X. Let LIM be a Banach limit and let $y \in C$ be such that $LIM_n\|x_n - y\|^2 = \inf_{x\in C} LIM_n\|x_n - x\|^2$. Then $LIM_n\langle x - y, J(x_n - y)\rangle \leq 0$ for all $x \in C$.*

Lemma 2.8 (cf. [29, Lemma 13.1]) *Let C be a convex subset of a smooth Banach space X, D be a nonempty subset of C and Q be a retraction from C onto D. Then the following are equivalent,*

(a) *Q is a sunny and nonexpansive.*
(b) *$\langle x - Qx, J(z - Qx)\rangle \leq 0$ for all $x \in C, z \in D$.*
(c) *$\langle x - y, J(Qx - Qy)\rangle \geq \|Qx - Qy\|^2$ for all $x, y \in C$.*

Lemma 2.9 (cf. [44]) *Let $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\varepsilon_n\}$ be sequences of nonnegative numbers satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \varepsilon_n/\alpha_n = 0$. Assume*

$$\gamma_n^2 \leq \gamma_{n-1}^2 - \alpha_n\psi(\gamma_n) + \varepsilon_n, \quad \forall n \in \mathbb{N},$$

where $\psi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that it is positive on $(0,\infty)$ and $\psi(0) = 0$. Then $\lim_{n\to\infty} \gamma_n = 0$.

3 Main results

First, we prove a strong convergence result of Browder's type for a family $\mathcal{T} = \{T_t : t > 0\}$ of nonexpansive mappings in a Banach space.

Theorem 3.1 *Let X be a reflexive Banach space X with a uniformly Gâteaux differentiable norm and let C be a nonempty, closed and convex subset of X. Let $\mathcal{T} = \{T_t : t > 0\}$ be a family of nonexpansive mappings from C into itself such that $F(\mathcal{T}) \neq \emptyset$. Let $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings from C into itself such that $\mathcal{T}$ has the property $(\mathscr{A})$ with respect to the family $\mathcal{S}$ and $F(\mathcal{T}) \subseteq F(\mathcal{S})$. Let $\{b_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} b_n = 0$. Assume that the family $\mathcal{T}$ has common fixed point in every weakly compact, convex and $\mathcal{T}-$invariant subset of C. Then we have the following,*

(a) *for each $u \in C$, the unique fixed point $y_n \in C$ of the contraction $C \ni y \mapsto b_n u + (1 - b_n)S_n y$ converges strongly to $Qu \in F(\mathcal{T})$ as $n \to \infty$;*
(b) *the mapping Q is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$.*

Proof (a) Let u be an element in C. From the Banach contraction principle, for every $n \in \mathbb{N}$, the unique fixed point $y_n \in C$ of the contraction $C \ni y \mapsto b_n u + (1 - b_n)S_n y$, is defined by

$$y_n = b_n u + (1 - b_n)S_n y_n. \tag{3.1}$$

One can easily see that $\{y_n\}$ is bounded. Hence

$$\|y_n - S_n y_n\| = b_n\|y_n - u\| \to 0 \text{ as } n \to \infty.$$

Since $\mathcal{T}$ has property $(\mathscr{A})$ with respect to the family $\mathcal{S}$, it follows that $y_n - T_r y_n \to 0$ for all $r > 0$.

Define the function $\varphi : C \to \mathbb{R}^+$ by $\varphi(x) = LIM_n\|y_n - x\|^2$, $x \in C$ and set $M := \{y \in C : \varphi(y) = \inf_{x\in C} \varphi(x)\}$. Since X is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$ and φ is a continuous convex function. From [5, Theorem 1.2, p. 79] we have that M is nonempty. From [55, Theorem 1.3.2, p. 22] and the above considerations, we see that M is also closed, convex and bounded. Moreover, M is invariant under T_r for each $r > 0$, i.e., $T_r(M) \subset M$ for all $r > 0$. In fact, we have for each $y \in M$,

$$\varphi(T_r y) = LIM_n \|y_n - T_r y\|^2 \leq LIM_n \|T_r y_n - T_r y\|^2 \leq LIM_n \|y_n - y\|^2 = \varphi(y).$$

Noticing that the set M is weakly compact, convex and $\mathcal{T}-$invariant. By assumption, the family $\mathcal{T}$ has a common fixed point in M, that is, $F(\mathcal{T}) \cap M \neq \emptyset$. Let $y^* \in F(\mathcal{T}) \cap M$. From Lemma 2.7, we get

$$LIM_n \langle z - y^*, J(y_n - y^*)\rangle \leq 0 \quad \text{for all} \quad z \in C. \tag{3.2}$$

The proof now follows similarly as in [6, Theorem 3.1]. We sketch here the proof for the sake of completeness. Fix $p \in F(\mathcal{T})$. From (3.1), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \langle y_n - p, J(y_n - p)\rangle \\
&= \langle b_n(u - p) + (1 - b_n)(S_n y_n - p), J(y_n - p)\rangle \\
&\leq b_n \langle u - p, J(y_n - p)\rangle + (1 - b_n)\|y_n - p\|^2,
\end{aligned}$$

which implies that

$$\|y_n - p\|^2 \leq \langle u - p, J(y_n - p)\rangle. \tag{3.3}$$

From (3.2) and (3.3) with $p = y^*$, we get

$$LIM_n \|y_n - y^*\|^2 \leq LIM_n \langle u - y^*, J(y_n - y^*)\rangle \leq 0.$$

Then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y^*$. If we assume the existence of another subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such $y_{n_i} \to z^*$, then $z^* \in F(\mathcal{T})$ and hence from (3.3) we get

$$\|y^* - z^*\|^2 \leq \langle u - y^*, J(y^* - z^*)\rangle.$$

By repeating last steps for $\{y_{n_i}\}$, it follows that

$$\|z^* - y^*\|^2 \leq \langle u - z^*, J(z^* - y^*)\rangle.$$

Summing last inequalities yields $y^* = z^*$.

In a summary, we have shown that each cluster point of $\{y_n\}$, equals y^*. Define $Qu = \lim_{n\to\infty} y_n$. Then Qu is the unique solution the following variational inequality

$$\langle Qu - u, J(Qu - p)\rangle \leq 0 \quad \text{for all} \quad p \in F(\mathcal{T}).$$

(b) By applying Lemma 2.8, we obtain that Q is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$. □

Remark 3.2 We note that the assumption that the family $\mathcal{T}$ has a common fixed point in every convex, weakly compact and $\mathcal{T}-$invariant subset of X had been widely investigated in the past (see, e.g., [37]).

We now prove strong convergence of a new inexact iteration method is for an uncountable family of nonexpansive mappings.

Theorem 3.3 *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and let C be a nonempty, closed and convex subset of X. Let $\mathcal{T} = \{T_t : t > 0\}$ be a family of nonexpansive mappings from C into itself such that $F(\mathcal{T}) \neq \emptyset$. Let $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings from C into itself such that $\mathcal{T}$ has the property $(\mathscr{A})$ with respect to the family $\mathcal{S}$ and $F(\mathcal{T}) \subseteq F(\mathcal{S})$. Assume that the family $\mathcal{T}$ has a common fixed point in every weakly compact, convex and $\mathcal{T}-$invariant subset of C. For any $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} y_n = (1-\delta_n)u + \delta_n \mathcal{P}e_n, n \in \mathbb{N}, \\ x_{n+1} = (1-\lambda_n)x_n + \lambda_n S_n x_n + \lambda_n\theta_n(y_n - x_n), \end{cases} \tag{3.4}$$

where $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ are three sequences of real numbers in $(0, 1]$ with $\lambda_n(1+\theta_n) \leq 1$, $\{e_n\}$ is an error sequence in X and $\mathcal{P}$ is a retraction from X onto C. Assume that $\{\mathcal{P}e_n\}$ is bounded and that the following conditions hold,

(*C1*) $\lim_{n\to\infty} \theta_n = 0$;
(*C2*) $\lim_{n\to\infty} \lambda_n/\theta_n = 0$;
(*C3*) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$;
(*C4*) $\lim_{n\to\infty} |\theta_{n-1} - \theta_n|/(\lambda_n\theta_n^2) = 0$;
(*C5*) $\lim_{n\to\infty} \delta_n = 0$.

Define

$$E_u = \left\{ z \in C : z = \frac{\theta_n}{1+\theta_n}u + \frac{1}{1+\theta_n}S_n z, n \in \mathbb{N} \right\}. \tag{3.5}$$

Assume that the following condition holds:

(*C6*) $\lim_{n\to\infty} \mathscr{D}_{E_u}(S_n, S_{n-1})/(\lambda_n\theta_n^2) = 0$.

Then $\{x_n\}$ converges strongly to $Qu \in F(\mathcal{T})$, where Q is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$.

Proof Let $\{b_n\}$ be a sequence in $(0, 1/2]$ defined by $b_n = \frac{\theta_n}{1+\theta_n}$ for all $n \in \mathbb{N}$. Let $\{z_n\}$ be a sequence in C implicitly defined by

$$z_n = \left(1 - \frac{1}{1+\theta_n}\right)u + \left(\frac{1}{1+\theta_n}\right)S_n z_n \tag{3.6}$$

From Theorem 3.1 we know that there exists a sunny nonexpansive retraction $Q : C \to F(\mathcal{T})$ such that $z_n \to Q(u)$ as $n \to \infty$.

Now the idea underlying the present proof is inspired by [21]. Firstly, we prove that $\{x_n\}$ is bounded. Let $\overline{d} := \|Q(u) - u\|$ and $\sigma_n := \|x_n - Q(u)\|$. From (3.4), we have

$$\begin{aligned} \sigma_{n+1} &\leq (1-\lambda_n(1+\theta_n))\|x_n - Q(u)\| + \lambda_n\|S_n x_n - Q(u)\| + \lambda_n\theta_n\|y_n - Q(u)\| \\ &\leq (1-\lambda_n(1+\theta_n))\sigma_n + \lambda_n\|S_n x_n - S_n Q(u)\| + \lambda_n\theta_n((1-\delta_n)\overline{d} + \delta_n\|\mathcal{P}e_n - Q(u)\|) \\ &\leq (1-\lambda_n\theta_n)\sigma_n + \lambda_n\theta_n(\overline{d} + \delta_n\|\mathcal{P}e_n - Q(u)\|). \end{aligned}$$

From the boundedness of $\{\mathcal{P}e_n - Q(u)\}$, there is a constant $K_1 \geq 0$ such that $\|\mathcal{P}e_n - Q(u)\| \leq K_1$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} \sigma_{n+1} &\leq (1-\lambda_n\theta_n)\sigma_n + \lambda_n\theta_n(\overline{d} + K_1) \\ &\leq \max\{\sigma_n, \overline{d} + K_1\} \\ &\leq \max\{\sigma_1, \overline{d} + K_1\}. \end{aligned}$$

Since $\{x_n\}$, $\{S_n x_n\}$ and $\{y_n\}$ are bounded, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1 - \lambda_n(1 + \theta_n))x_n - x_n + \lambda_n S_n x_n + \lambda_n \theta_n y_n\| \\ &\leq \lambda_n(\|S_n x_n\| + (1 + \theta_n)\|x_n\| + \theta_n\|y_n\|) \\ &\leq \lambda_n K_2 \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$ and for some $K_2 > 0$. From (3.7), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Now, we will prove that $\{x_n\}$ converges to $Q(u)$. From (3.6) that

$$\theta_n(u - z_n) = z_n - S_n z_n. \tag{3.8}$$

Set $B := \{z_n\}$, $\gamma_n = \|x_{n+1} - z_n\|$ and

$$\begin{aligned}\varepsilon_n &= \|z_n - z_{n-1}\|(2\|x_n - z_{n-1}\| + \|z_n - z_{n-1}\|) + 2\lambda_n\theta_n\|x_{n+1} - x_n\|\|x_{n+1} - z_n\| \\ &\quad + 2\lambda_n\theta_n\delta_n\|u - \mathcal{P}e_n\|\|x_{n+1} - z_n\| + 4\lambda_n\|x_{n+1} - x_n\|\|x_{n+1} - z_n\|.\end{aligned}$$

We now claim that

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\lambda_n\theta_n} = 0.$$

We infer that

$$\begin{aligned}\frac{\varepsilon_n}{\lambda_n\theta_n} &= \frac{\|z_n - z_{n-1}\|}{\lambda_n\theta_n}(2\|x_n - z_{n-1}\| + \|z_n - z_{n-1}\|) + 2\|x_{n+1} - x_n\|\|x_{n+1} - z_n\| \\ &\quad + 2\delta_n\|u - \mathcal{P}e_n\|\|x_{n+1} - z_n\| + 4\frac{\|x_{n+1} - x_n\|\|x_{n+1} - z_n\|}{\theta_n}.\end{aligned} \tag{3.9}$$

Since S_n is nonexpansive, it is well known that $I - S_n$ is an accretive operator, i.e., for any x, y and $r > 0$ it holds

$$\|x - y\| \leq \|x - y + \frac{1}{r}((I - S_n)x - (I - S_n)y)\|.$$

Then we have

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq \left\| z_n - z_{n-1} + \frac{1}{\theta_n}((z_n - S_n z_n) - (z_{n-1} - S_n z_{n-1}))\right\| \\ &\leq \left\| z_n - z_{n-1} + \frac{1}{\theta_n}((z_n - S_n z_n) - (z_{n-1} - S_{n-1} z_{n-1}))\right\| \\ &\quad + \frac{1}{\theta_n}\|S_n z_{n-1} - S_{n-1} z_{n-1}\|.\end{aligned}$$

It follows from (3.8) that

$$\|z_n - z_{n-1}\| \leq \frac{|\theta_{n-1} - \theta_n|}{\theta_n}\|z_{n-1} - u\| + \frac{1}{\theta_n}\mathscr{D}_B(S_n, S_{n-1}).$$

We get from $\lim_{n\to\infty} \frac{|\theta_{n-1} - \theta_n|}{\lambda_n\theta_n^2} = 0$ and Condition $(C6)$ that

$$\lim_{n\to\infty} \frac{\|z_n - z_{n-1}\|}{\lambda_n\theta_n} = 0.$$

Moreover, from (3.7) we obtain

$$\frac{\|x_{n+1}-x_n\|\|x_{n+1}-z_n\|}{\theta_n} \leq K_2 \sup_{k\in\mathbb{N}} \|x_{k+1}-z_k\| \frac{\lambda_n}{\theta_n} \to 0 \text{ as } n \to \infty,$$

since $\lim_{n\to\infty} \lambda_n/\theta_n = 0$. Hence, from (3.9), we obtain

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\lambda_n\theta_n} = 0.$$

Note that

$$y_n = (1-\delta_n)u + \delta_n \mathcal{P}e_n,$$

and hence from (3.8) we get

$$\theta_n(y_n - z_n) - (z_n - S_n z_n) = \theta_n\delta_n(\mathcal{P}e_n - u).$$

From Lemma 2.6 we have

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &= \|(1-\lambda_n)x_n + \lambda_n S_n x_n + \lambda_n\theta_n(y_n - x_n) - z_n\|^2 \\ &\leq \|x_n - z_n\|^2 + 2\langle(1-\lambda_n)x_n + \lambda_n S_n x_n \\ &\quad + \lambda_n\theta_n(y_n - x_n) - x_n, J(x_{n+1}-z_n)\rangle \\ &= \|x_n - z_n\|^2 + 2\langle\lambda_n\theta_n(z_n - x_{n+1} + x_{n+1} - z_n) - \lambda_n x_n \\ &\quad + \lambda_n S_n x_n + \lambda_n\theta_n(y_n - x_n), J(x_{n+1}-z_n)\rangle\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &\leq \|x_n - z_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-z_n\|^2 \\ &\quad + 2\lambda_n\langle\theta_n(x_{n+1}-z_n) - (x_n - S_n x_n) \\ &\quad + \theta_n(y_n - x_n), J(x_{n+1}-z_n)\rangle \\ &= \|x_n - z_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-z_n\|^2 \\ &\quad + 2\lambda_n\langle\theta_n(x_{n+1}-x_n) + \theta_n(y_n - z_n) - (z_n - S_n z_n) \\ &\quad + (z_n - S_n z_n) - (x_{n+1} - S_n x_{n+1}) \\ &\quad + (x_{n+1} - S_n x_{n+1}) - (x_n - S_n x_n), J(x_{n+1}-z_n)\rangle.\end{aligned}$$

From the nonexpansivity of S_n, we obtain

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &\leq \|x_n - z_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-z_n\|^2 \\ &\quad + 2\lambda_n\theta_n\|x_{n+1}-x_n\|\|x_{n+1}-z_n\| \\ &\quad + 2\lambda_n\theta_n\delta_n\|u - \mathcal{P}e_n\|\|x_{n+1}-z_n\| \\ &\quad + 4\lambda_n\|x_{n+1}-x_n\|\|x_{n+1}-z_n\|,\end{aligned}$$

which leads to

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &\leq (\|x_n - z_{n-1}\| + \|z_{n-1} - z_n\|)^2 - 2\lambda_n\theta_n\|x_{n+1}-z_n\|^2 \\ &\quad + 2\lambda_n\theta_n\|x_{n+1}-x_n\|\|x_{n+1}-z_n\| \\ &\quad + 2\lambda_n\theta_n\delta_n\|u - \mathcal{P}e_n\|\|x_{n+1}-z_n\| \\ &\quad + 4\lambda_n\|x_{n+1}-x_n\|\|x_{n+1}-z_n\|.\end{aligned}$$

The previous inequality can be written as

$$\gamma_n^2 \leq \gamma_{n-1}^2 - 2\lambda_n\theta_n\gamma_n^2 + \varepsilon_n.$$

Now, from Lemma 2.9, we deduce

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} \gamma_n = 0.$$

From Theorem 3.1, we know that $z_n \to Q(u)$ as $n \to \infty$ and so $x_n \to Q(u)$ as required. □

Corollary 3.4 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty, closed and convex subset of X which has the fixed-point property for nonexpansive mappings and let $\mathcal{P}$ be a retraction from X onto C. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ be three sequences of real numbers in $(0, 1]$ with $\lambda_n(1+\theta_n) \leq 1$ satisfying Conditions (C1)–(C5) of Theorem 3.3. For any $u, x_1 \in X$, let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = (1 - \lambda_n(1+\theta_n))x_n + \lambda_n T x_n + \lambda_n\theta_n((1-\delta_n)u + \delta_n \mathcal{P} e_n),$$

where $\{e_n\}$ is an error sequence in X such that $\{\mathcal{P} e_n\}$ is bounded. Then $\{x_n\}$ converges strongly to $Qu \in F(T)$, where Q is a sunny nonexpansive retraction from C onto $F(T)$.

Now, let X be a strictly convex Banach space and let $\mathcal{T} = \{T_n\}$ be an infinite family of nonexpansive mappings from a nonempty, closed and convex subset C of X into itself with $F(\mathcal{T}) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ be three sequences of real numbers in $(0, 1]$ with $\lambda_n(1+\theta_n) \leq 1$ satisfying condition conditions (C1)–(C5) and let $\{\beta_{n,k}\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that $\sum_{k=1}^{n} \beta_{n,k} = 1$ for each $n \in \mathbb{N}$.

We now construct an auxiliary sequence of nonexpansive mappings $\mathcal{S} = \{S_n\}$ as follows: For $x \in C$,

$$S_n x = \sum_{k=1}^{n} \beta_{n,k} T_k x, \quad n \in \mathbb{N}. \tag{3.10}$$

Then from the condition $\sum_{k=1}^{n} \beta_{n,k} = 1$, each S_n is also a nonexpansive mapping of C into itself such that $\cap_{k\in\mathbb{N}} F(T_k) \subseteq \cap_{n\in\mathbb{N}} F(S_n)$.

Theorem 3.5 *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable and let C be a nonempty, closed and convex subset of X. Let $\mathcal{T} = \{T_n\}$ be an infinite family of nonexpansive mappings from C into itself with $F(\mathcal{T}) \neq \emptyset$. Assume that the family $\mathcal{T}$ has a common fixed point in every weakly compact, convex and $\mathcal{T}-$invariant subset of C. Let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ be three sequences of real numbers in $(0, 1]$ with $\lambda_n(1+\theta_n) \leq 1$ satisfying Conditions (C1)–(C5) of Theorem 3.3 and let $\{\beta_{n,k}\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that*

- *(B1)* $\sum_{k=1}^{n} \beta_{n,k} = 1$ *for each* $n \in \mathbb{N}$;
- *(B2)* $\lim_{n\to\infty} \frac{1}{\lambda_n\theta_n^2} \sum_{k=1}^{n-1} |\beta_{n-1,k} - \beta_{n,k}| = 0.$

Suppose that $\mathcal{S} = \{S_n\}$ be a sequence of nonexpansive mappings defined in (3.10) and that $\mathcal{T}$ has the property ($\mathscr{A}$) with respect to $\mathcal{S}$. For any $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (3.4), where $\{e_n\}$ is an error sequence in X and $\mathcal{P}$ is a retraction from X onto C. Assume that $\{\mathcal{P} e_n\}$ is bounded. Then $\{x_n\}$ converges strongly to $Qu \in F(\mathcal{T})$, where Q is a sunny nonexpansive retraction from C onto $F(\mathcal{T})$.

Proof Noticing that the set E_u defined by (3.5) is in $\mathscr{B}(C)$. Let $K^* = \sup\{\|T_n z\| : z \in E_u \text{ and } n \in \mathbb{N}\}$. By using argument of [3], we see that

$$\mathscr{D}_{E_u}(S_n, S_{n+1}) \leq 2 \sum_{k=1}^{n} \left|\beta_{n,k} - \beta_{n+1,k}\right| K^*.$$

From the condition (B2), we obtain $\lim_{n\to\infty} \mathscr{D}_{E_u}(S_n, S_{n-1})/(\lambda_n \theta_n^2) = 0$. Therefore, Theorem 3.5 follows from Theorem 3.3. □

Remark 3.6 In view of [12], some convergence theorems are established in [3,51] under the following assumptions:

(i) $\sum_{k=1}^{n} \beta_{n,k} = 1$ for each $n \in \mathbb{N}$;
(ii) $\lim_{n\to\infty} \beta_{n,k} > 0$ for each $k \in \mathbb{N}$;
(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1,k} - \beta_{n,k}| < \infty$.

4 Applications

4.1 Zeros of accretive operators

In the general setting of Banach spaces, let $A \subset X \times X$ be an operator on the Banach space X. The set $D(A)$ defined by $D(A) = \{x \in X : Ax \neq \emptyset\}$ is called the domain of A, the set $R(A)$ defined by $R(A) = \bigcup_{x\in X} Ax$ is called the range of A and the set $G(A)$ defined by $G(A) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\}$ is called the graph of A. A zero (or root) of A is a point $z \in D(A)$ such that $0 \in Az$. We denote by $A^{-1}0$ the zeros set of A. An operator $A \subset X \times X$ is said to be

(1) *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$;
(2) *maximal accretive* if it is accretive and the inclusion $G(A) \subseteq G(B)$, with B accretive, implies $G(A) = G(B)$;
(3) *m-accretive* if A is accretive and $R(I + \mu A) = X$ for any $\mu > 0$.

If A is an *m-accretive* operator, A has no proper accretive extension. However, not every maximal accretive operator is *m-accretive*.

An accretive operator A defined on a Banach space X is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. If A is an accretive operator, the resolvent of A defined for each $\lambda > 0$, as the nonexpansive single-valued mapping $J_\lambda^A : R(I+\lambda A) \to D(A)$ by $J_\lambda^A = (I+\lambda A)^{-1}$. It is well known that for an accretive operator A which satisfies the range condition, $A^{-1}(0) = F(J_\lambda^A)$ for all $\lambda > 0$. The Yosida approximation A_r defined by $A_r = (I - J_r^A)/r$. We know that $A_r x \in A J_r^A x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq |Ax| = \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$.

Accretive operators were introduced independently in 1967 by Browder [11] and Kato [36]. Interest in such maps stems mainly from their firm connection with evolution equations. It is known (see, e.g., [62]) that many physically significant problems can be modeled by initial-value problems of the form

$$x'(t) + Ax(t) = 0, x(0) = x_0, \tag{4.1}$$

where $A : X \to X$ is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schřodinger equations. The solutions of the problem

$$\text{find } z \in X \quad \text{such that} \quad Az = 0$$

are precisely the equilibrium points of the system (4.1). Consequently, considerable research efforts have been devoted, especially within the past 25 years or so, to iterative methods for approximating these equilibrium points.

In case of a Hilbert space H, accretive operators are also called monotone. An interesting fact is that a monotone operator A on H is maximal if and only if $R(I+A) = H$. This was originally due to Minty [43] who provided a crucial characterization of maximal monotone operators. The best-known example of maximal monotone operator is the subgradient mapping $\partial\psi$ of a closed, proper and convex function $\psi : H \to \mathbb{R} \cup \{\infty\}$ (see for instance, [48]). In the case that A is the subdifferential map $\partial\psi$ of a convex function ψ, $A^{-1}0$ is the set of all global minima of ψ. In this paper, we study a more general situation.

Given an accretive operator $A \subset X \times X$, we consider the following problem

$$\text{find } z \in X \quad \text{such that} \quad 0 \in Az. \tag{P}$$

The Problem (P) can be regarded as a unified formulation of several important problems. For an appropriate choice of the operator A, Problem (P) covers a wide range of mathematical applications; for example, variational inequalities, complementarity problems and non-smooth convex optimization. Problem (P) has applications in physics, economics and in several areas of engineering. Therefore, one of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of maximal monotone operators. One method for finding zeros of maximal monotone operators is the *proximal point algorithm*. Let A be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = J_{r_n}^{A} x_n \tag{4.2}$$

where $\{r_n\}$ is a regularization sequence in $(0, \infty)$. Note that (4.2) is equivalent to

$$0 \in \frac{1}{r_n}(x_{n+1} - x_n) + Ax_{n+1}.$$

This method was first introduced by Martinet [42]. If $\psi : H \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semicontinuous and convex function, then the algorithm reduces to

$$x_{n+1} = \operatorname{argmin}_{y \in H}\{\psi(y) + \frac{1}{2r_n}\|x_n - y\|^2\}$$

Rockafellar [49] studied the proximal point algorithm in the framework of Hilbert space and he proved the following result.

Theorem 4.1 *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let $\{x_n\}$ be a sequence in H defined by (4.2), where $\{r_n\}$ is a sequence in $(0, \infty)$ such that $\liminf_{n\to\infty} r_n > 0$. If $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $A^{-1}0$.*

Güler [31] constructed a counterexample showing that the sequence generated by (4.2) does not converge strongly, in general. This brings us a natural question on how to modify the proximal point algorithm so that strongly convergence is guaranteed. Recently, motivated by Halpern [33], some authors (see, for examples Benavides, Acedo and Xu [7], Kamimura and Takahashi [35], Mainge [39], Marino and Xu [41], Takahashi and Ueda [54] and Nakajo [45]) modified the proximal point algorithm to generate strongly convergent sequences.

However, as pointed out in Eckstein [26], this ideal form of the proximal point method is often impractical, since in many cases the exact iteration (4.2) may require a computation as difficult as solving the original Problem (P). Rockafellar [49] has given a more practical method which is an inexact variant of the method

$$e_n \in x_n - x_{n-1} + r_n A x_n,$$

where $\{e_n\}$ is an error sequence. The method is called inexact proximal point algorithm. It was shown in Rockafellar [49] that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in A(z)$. In 2002, Xu [57] modified the proximal point algorithm for solving Problem (P) and proved strong convergence of the algorithm in a Hilbert space under the same assumption $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

The criteria $\sum_{n=1}^{\infty} \|e_n\| < \infty$ imposed for convergence of inexact proximal point algorithms is somewhat undesirable, because it impose increasing precision along the iterative process. This brings us to the following natural question,

Question 4.2 *Is it possible to further modify inexact proximal point algorithm without the assumption $\sum_{n=1}^{\infty} \|e_n\| < \infty$, so that it can generate a strongly convergent sequence in Banach space setting?*

Recently, Sahu and Yao [53] introduced and studied the prox-Tikhonov method for solving Problem (P) in Banach space and they partially answered Question 4.2.

Before pressing our results, we need the following:

Proposition 4.3 *(cf. [55]) Let X be a Banach space X and let $A \subset X \times X$ be an accretive operator satisfying the range condition, i.e., $\overline{D(A)} \subset R(I + tA)$ for all $t > 0$. Then*

$$\frac{1}{r}\|J_t x - J_r^A J_t^A x\| \leq \frac{1}{t}\|x - J_t^A x\| \quad for\ all \quad x \in R(I + rA) \quad and \quad r, t > 0.$$

Lemma 4.4 *(cf. [56, Corollary 3.4]) Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty, closed and convex subset of X and let $A \subset X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Suppose that every closed, convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Then we have the following*

- (a) *for each $x \in C$, $\{J_t x\}$ converges strongly to Qx as $t \to \infty$, where Q is the sunny nonexpansive retraction from C onto $A^{-1}0$,*
- (b) *the set $A^{-1}0$ is a sunny nonexpansive retract of C.*

The following proposition shows that for an accretive operator A, the family $\{J_t^A : t > 0\}$ of resolvent operators enjoys property $(\mathscr{A})$.

Proposition 4.5 *Let C be a nonempty, closed and convex subset of a Banach space X and let $A \subset X \times X$ be an accretive operator such that $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. Then:*

- (a) *The family $\{J_t^A : t > 0\}$ of resolvent operators enjoys property $(\mathscr{A})$.*
- (b) *If $\{t_n\}$ is a sequence in $(0, \infty)$ such that $\inf\{t_n : n \in \mathbb{N}\} > 0$, then the family $\{J_t^A : t > 0\}$ of resolvent operators enjoys property $(\mathscr{A})$ with respect to the sequence $\{J_{t_n}^A : n \in \mathbb{N}\}$.*

(*c*) *For* $\lambda, \mu > 0$ *and* $B \in \mathscr{B}(C)$*, it holds:*

$$\mathscr{D}_B(J_\lambda^A, J_\mu^A) \leq \frac{|\lambda - \mu|}{\lambda} \sup_{x \in B} \|x - J_\lambda^A x\| \quad \text{for all} \quad x \in B.$$

Proof (a) Let $r > 0$ and let $\{z_t\}_{t>0} \in \mathscr{B}(C)$ such that $\|z_t - J_t^A z_t\| \to 0$ as $t \to \infty$. From Proposition 4.3, we have

$$\frac{1}{r}\|J_t^A z_t - J_r^A J_t^A z_t\| \leq \frac{1}{t}\|z_t - J_t^A z_t\|. \tag{4.3}$$

Using (4.3), we have

$$\begin{aligned}\|z_t - J_r^A z_t\| &\leq \|z_t - J_t^A z_t\| + \|J_t^A z_t - J_r^A J_t^A z_t\| + \|J_r^A J_t^A z_t - J_r^A z_t\| \\ &\leq \left(1 + \frac{r}{t}\right)\|z_t - J_t^A z_t\| + \|J_t^A z_t - z_t\| \to 0 \text{ as } t \to \infty.\end{aligned}$$

(b) Assume that $\{t_n\}$ is a sequence in $(0, \infty)$ such that $\inf\{t_n : n \in \mathbb{N}\} > 0$. Let $r > 0$ and let $\{z_n\}$ be a bounded sequence in C such that $\|z_n - J_{t_n}^A z_n\| \to 0$ as $n \to \infty$. From Proposition 4.3, we have

$$\begin{aligned}\|z_n - J_r^A z_n\| &\leq \|z_n - J_{t_n}^A z_n\| + \|J_{t_n}^A z_n - J_r^A J_{t_n}^A z_n\| + \|J_r^A J_{t_n}^A z_n - J_r^A z_n\| \\ &\leq 2\|z_n - J_{t_n}^A z_n\| + \|J_{t_n}^A z_n - J_r^A J_{t_n}^A z_n\| \\ &\leq 2\|z_n - J_{t_n}^A z_n\| + \frac{r}{t_n}\|z_n - J_{t_n}^A z_n\|.\end{aligned}$$

Since $\inf\{t_n : n \in \mathbb{N}\} > 0$ and $\|z_n - J_{t_n}^A z_n\| \to 0$ as $n \to \infty$, we obtain $\|z_n - J_r^A z_n\| \to 0$ as $n \to \infty$.

(c) Let $\lambda, \mu > 0$ and $B \in \mathscr{B}(C)$. From [3,55] and $x \in B$, we have

$$\|J_\lambda^A x - J_\mu^A x\| \leq \frac{|\lambda - \mu|}{\lambda}\|x - J_\lambda^A x\|. \tag{4.4}$$

It follows from (4.4), we have

$$\mathscr{D}_B(J_\lambda^A, J_\mu^A) \leq \frac{|\lambda - \mu|}{\lambda} \sup_{x \in B} \|x - J_\lambda^A x\|.$$

□

Theorem 4.6 *Let* X *be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let* C *be a closed and convex subset of* X*. Let* $A \subset X \times X$ *be an accretive operator with resolvent* J_t^A *for* $t > 0$ *such that* $A^{-1}0 \neq \emptyset$ *and* $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$*. For* $t > 0$*, assume that* J_t^A *has a fixed point in every weakly compact, convex and* J_t^A*-invariant subset of* C*. Then we have the following*

(*a*) *for each fixed* $u \in C$*, the unique fixed point* $y_n \in C$ *of the contraction* $C \ni y \mapsto b_n u + (1 - b_n)J_{t_n}^A y$ *converges strongly as* $n \to \infty$ *to* $Q(u)$*, where* Q *is a sunny nonexpansive retraction from* C *onto* $A^{-1}0$*,* $\{t_n\}$ *is a sequence in* $(0, \infty)$ *and* $\{b_n\}$ *is a sequence in* $(0, 1)$ *with* $b_n \to 0$ *as* $n \to \infty$*.*

(*b*) *The set* $A^{-1}0$ *is a sunny nonexpansive retract of* C*.*

Proof Theorem 4.6 follows from Theorem 3.1. □

Remark 4.7 The conclusion of Theorem 4.6 slightly differs from that of Lemma 4.4.

Theorem 4.8 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty, closed and convex subset of X and let $\mathcal{P}$ be a retraction from X onto C. Let $A \subset X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$. For $t > 0$, assume that J_t^A has a fixed point in every weakly compact, convex and J_t^A-invariant subset of C. Let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ be three sequences of real numbers in $(0, 1]$ with $\lambda_n(1 + \theta_n) \leq 1$ satisfying the Conditions (C1)–(C5) of Theorem 3.3. For any $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} y_n &= (1-\delta_n)u + \delta_n \mathcal{P} e_n, n \in \mathbb{N}, \\ x_{n+1} &= (1-\lambda_n)x_n + \lambda_n J_{t_n}^A x_n + \lambda_n \theta_n (y_n - x_n), \end{cases}$$

where $\{e_n\}$ is an error sequence in X and $\{t_n\}$ is a sequence in $(0, \infty)$ such that $\inf\{t_n : n \in \mathbb{N}\} > 0$ and $\lim_{n\to\infty} |t_n - t_{n-1}|/(\lambda_n \theta_n^2) = 0$. Assume that $\{\mathcal{P} e_n\}$ is bounded. Then $\{x_n\}$ converges strongly to $Qu \in A^{-1}0$, where Q is a sunny nonexpansive retraction from C onto $A^{-1}0$.

Proof Let $\{b_n\}$ be a sequence in $(0, 1/2]$ defined by $b_n = \frac{\theta_n}{1+\theta_n}$ for all $n \in \mathbb{N}$ and let $\{z_n\}$ be a sequence in C defined by $z_n = (1 - b_n)u + b_n J_{t_n}^A z_n$. Proposition 4.5 (b) implies that the family $\mathcal{J}^A := \{J_t^A : t > 0\}$ satisfies the property $(\mathscr{A})$ with respect $\mathcal{S}^A := \{J_{t_n}^A : n \in \mathbb{N}\}$. Noticing that $F(\mathcal{J}^A) = F(\mathcal{S}^A) = A^{-1}0$. It follows from Theorem 3.1 that there exists a sunny nonexpansive retraction $Q : C \to A^{-1}0$ such that $z_n \to Q(u)$ as $n \to \infty$. Set $S_n := J_{t_n}^A$ and $B = \{z_n\}$. We now show that the Condition (C6) holds. From Proposition 4.5 (c), we get that

$$\begin{aligned} \mathscr{D}_B(S_n, S_{n-1}) &\leq \frac{|t_n - t_{n-1}|}{t_n} \sup_{z_n \in B} \|z_n - J_{t_{n-1}}^A z_n\| \\ &\leq \frac{|t_n - t_{n-1}|}{t_n} \sup_{z_n \in B} (\|z_n - Q(u)\| + \|J_{t_{n-1}}^A z_n - Q(u)\|) \\ &\leq \frac{|t_n - t_{n-1}|}{t_n} \sup_{k \in \mathbb{N}} (2\|z_k - Q(u)\|). \end{aligned}$$

By assumption, we conclude that $\mathscr{D}_B(S_n, S_{n-1})/(\lambda_n \theta_n^2) \to 0$ as $n \to \infty$. Therefore, Theorem 4.8 follows from Theorem 3.3. □

Remark 4.9 An example of the sequences $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ which satisfy the hypotheses of Theorem 4.8 is

$$\lambda_n = \frac{1}{n^{2/3}}, \ \theta_n = \frac{1}{n^{1/6}} \quad \text{and} \quad \delta_n = \frac{1}{n}.$$

Remark 4.10 (i) Theorem 4.8 provides an algorithm for finding solutions of the problem:

$$\text{find } z \in X \text{ such that } 0 \in A(z) \cap C.$$

(ii) We suspect, although we have no proof, that property $(\mathscr{A})$ for the family $\{S_n\}$ is granted by the fact that $\{J_t^A\}$ does satisfy such property.

(iii) The assumption "acceptably" paired for the sequences $\{\lambda_n\}$ and $\{\theta_n\}$ are used the for finding zeros of accretive operators in [9,13,50]. In our results the sequences $\{\lambda_n\}$ and $\{\theta_n\}$ are not necessarily acceptably paired.

4.2 Minimization

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $\psi : H \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous and convex function, where H is an Hilbert space. The subdifferential $\partial \psi$ of ψ is defined by

$$\partial \psi(z) = \{w \in H : f(y) \geq f(z) + \langle y - z, w \rangle, y \in H\}$$

If $A = \partial \psi$, then A is a maximal monotone operator (see, Rockafellar [48, Theorem 4]). We also know that

(i) $0 \in Av$ if and only if $v = \operatorname{argmin}_{z \in H} f(z)$,
(ii) $J_t^A x = \operatorname{argmin}_{z \in H}\{\psi(z) + \|z - x\|^2/2t\}$ for all $t > 0$ and $x \in H$.

As direct a consequence of Theorem 3.1, we obtain the following result.

Theorem 4.11 *Let H be a Hilbert space and let $\psi : H \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous and convex function with $(\partial \psi)^{-1} 0 \neq \emptyset$. Let $A = \partial \psi$. Then for each fixed $u \in H$ and any sequence $\{b_n\}$ in $(0, 1)$ such that $b_n \to 0$ as $n \to \infty$, the sequence $\{y_n\}$ defined as*

$$y_n = b_n u + (1 - b_n) z_n,$$

where $\{t_n\}$ is a sequence in $(0, \infty)$ and $z_n = \operatorname{argmin}_{z \in H}\{\psi(z) + \|z - y_n\|^2/2t_n\}$ for all $n \in \mathbb{N}$, strongly converges to $P_{A^{-1}(0)}(u)$.

4.3 Split feasibility problems

Let C be a nonempty, closed and convex subset of real Hilbert space H and let $A : C \to H$ be a nonlinear operator. We say that A is

(a) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2 \quad \text{for all} \quad x, y \in C,$$

(b) ν-inverse strongly monotone (ν-ism) if there exists a constant $\nu > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \nu \|Ax - Ay\|^2 \quad \text{for all} \quad x, y \in C.$$

Recall that a mapping T in a Hilbert space H is said to be averaged if T can be written as $(1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and S is nonexpansive on H.
Construction of fixed points of nonexpansive operators is an important subject in the theory of nonexpansive operators and its applications in a number of applied areas, in particular, in image recovery and signal processing (see, e.g., [15,47,60,61]).
For instance, split feasibility problem (SFP) was first introduced by Censor and Elfving [16] in finite dimensional Hilbert spaces and finds applications in medical image reconstruction [14] and to model the intensity-modulated radiation therapy [17–20]. Iterative algorithms for approaching solutions for the SFP in infinite-dimensional Hilbert spaces, had been recently studied by Xu in [59].
The SFP is formulated as follows:

$$\text{find a point } x \in C \quad \text{such that} \quad Ax \in Q, \tag{4.5}$$

here C is a closed and convex subset of a Hilbert space H_1, Q is a closed and convex subset of another Hilbert space H_2 and $A : H_1 \to H_2$ is a bounded linear operator. The

SFP is said to be consistent if (4.5) has a solution. It is easy to see that SFP is consistent if and only if the following fixed point problem has a solution:

$$\text{find } x \in C \quad \text{such that} \quad x = P_C(I - \gamma A^*(I - P_Q)A)x, \tag{4.6}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively; $\gamma > 0$, and A^* is the adjoint of A. Note that for sufficient small $\gamma > 0$, the operator $P_C(I - \gamma A^*(I - P_Q)A)$ is nonexpansive. Set $q(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2$ for all $x \in C$.
Consider the minimization problem:

$$\text{find } z \in C \quad \text{such that} \quad z \in \min_{x \in C} q(x).$$

From [4], the gradient of q is

$$\nabla q = A^*(I - P_Q)A.$$

Since $I - P_Q$ is nonexpansive, it follows that ∇q is L-Lipschitzian with $L = \|A\|^2$. Therefore, ∇q is $1/L$ − ism (cf. [4]) and for any $0 < \gamma < 2/L$, $I - \gamma\nabla q$ is averaged. Therefore, the composite $P_C(I - \nabla q)$ is also averaged. Set $T := P_C(I - \nabla q)$. Note that solution set of SFP(4.5) is denoted by $F(T)$.

It is well known that the sequence $\{T^n x\}$ of iterates of nonexpansive operator T at a point $x \in C$ may, in general, not behave well. This means that it may not converge (even in the weak topology). One way to overcome this difficulty is to use the Krasnoselskii-Mann (KM) iteration method (see, [15]) that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where the initial guess $x_1 \in C$ is chosen arbitrarily. It is worth noting that the KM iteration process is a well known process for finding fixed points of nonexpansive operators (see, [15]) and it is further developed in a general context in [58]. See also [52] for improved convergence results.

We now apply the result established in Sect. 3 for finding solutions of SFP (4.5).

Theorem 4.12 *Assume that SFP(4.5) is consistent. Let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ be three sequences in $(0, 1]$ with $\lambda_n(1 + \theta_n) \le 1$ satisfying the Conditions (C1)–(C5) of Theorem 3.3. For any $u, x_1 \in H$, let $\{x_n\}$ be a sequence in H generated by*

$$x_{n+1} = (1 - \lambda_n(1 + \theta_n))x_n + \lambda_n P_C(I - \gamma\nabla q)x_n + \lambda_n\theta_n((1 + \delta_n)u + \delta_n e_n),\ n \in \mathbb{N},$$

where $0 < \gamma < 2/L$ and $\{e_n\}$ is a bounded error sequence in H. Then $\{x_n\}$ converges strongly to the solution of SFP(4.5) nearest to u.

Proof Since $T := P_C(I - \lambda\nabla q)$ is nonexpansive, Theorem 4.12 follows from Corollary 3.4. □

4.4 Equilibrium problems

Let H be a Hilbert space and let $G : H \times H \to \mathbb{R}$ be an equilibrium function, that is

$$G(u, u) = 0 \quad \text{for every} \quad u \in H.$$

The equilibrium problem is defined by

$$\text{find } \tilde{x} \in H \quad \text{such that} \quad G(\tilde{x}, y) \ge 0 \quad \text{for all} \quad y \in H.$$

A solution $\tilde{x}$ of the equilibrium problem is called an equilibrium point and the set of all equilibrium points will be denoted by $EP(G)$. The topic has been considered by several authors (see, for instance, [2,8–10,28,46]) with the purpose of extending results concerning particular problems to more general settings. We assume some mild conditions over G in such a way that the results can be applied in several cases including optimization problems, fixed point problems, variational problems, variational inequality problems, and convex vector minimization problems [24,34].

Lemma 4.13 *(cf. [24]).Let C be a nonempty, closed and convex subset of H and assume that $G : C \times C \to \mathbb{R}$ satisfy*

- $(A1)$ *for all $x \in C$, $G(x,x) = 0$;*
- $(A2)$ *G is monotone, i.e. $G(x,y) + G(y,x) \leq 0$ for all $x, y \in C$;*
- $(A3)$ *for all $x, y, z \in C$,*

$$\limsup G(tz + (1-t)x, y) \leq G(x,y) \text{ as } t \to 0;$$

- $(A4)$ *for all $x \in C$, $y \mapsto G(x,y)$ is convex and lower semicontinuous.*

For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be the resolvent of G,

$$S_r(x) := \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\},$$

then S_r is well defined and the following hold:

(1) *S_r is single-valued;*
(2) *S_r is firmly nonexpansive, i.e.,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle,$$

for all $x, y \in H$;
(3) *$F(S_r) = EP(G)$;*
(4) *$EP(G)$ is closed and convex.*

Lemma 4.14 *(cf. [40]) For any fixed $x \in H$ and $t, r > 0$ it holds*

$$\|S_t x - S_r S_t x\| \leq \left(1 + \left|\frac{t-r}{t}\right|\right) \|x - S_t x\|.$$

From this, we deduce the property $(\mathscr{A})$.

Lemma 4.15 *Let G be an equilibrium function satisfying the assumptions of Lemma 4.13. Then the family $\{S_t : t > 0\}$ enjoys the property $(\mathscr{A})$.*

Proof Let $\{z_t\} \in \mathscr{B}(H)$ such that $z_t - S_t z_t \to 0$. Then, for any fixed $r > 0$,

$$\begin{aligned} \|z_t - S_r z_t\| &\leq \|z_t - S_t z_t\| + \|S_t z_t - S_r S_t z_t\| + \|S_r S_t z_t - S_r z_t\| \\ &\leq \left(3 + \left|\frac{t-r}{t}\right|\right) \|z_t - S_t z_t\| \end{aligned}$$

by the nonexpansivity and Lemma 4.14. In particular, we get

$$\lim_{t\to\infty} \|z_t - S_r z_t\| = 0.$$

□

Similarly, we have

Lemma 4.16 *Let G be an equilibrium function satisfying the assumptions of Lemma 4.13 and let $\{t_n\}$ be a sequence in $(0, \infty)$. Then the family $\{S_t : t > 0\}$ enjoys property $(\mathscr{A})$ with respect the sequence $\{S_{t_n} : n \in \mathbb{N}\}$.*

Theorem 4.17 *Let H be a Hilbert space. Let $G : H \times H \to \mathbb{R}$ be an equilibrium function satisfying the assumptions of Lemma 4.13. Let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\delta_n\}$ be three sequences of real numbers in $(0, 1]$ with $\lambda_n(1 + \theta_n) \leq 1$ satisfying the Conditions (C1)–(C5) of Theorem 3.3. Let $\{S_{t_n}\}$ be a family of resolvent operators for G, where $\{t_n\}$ is a sequence in $(0, \infty)$ such that $\inf\{t_n : n \in \mathbb{N}\} > 0$ and $\lim_{n\to\infty} |t_n - t_{n-1}|/(\lambda_n\theta_n^2) = 0$. For any $u, x_1 \in H$, let $\{x_n\}$ be a sequence in H generated by*

$$\begin{cases} y_n &= (1 - \delta_n)u + \delta_n e_n, n \in \mathbb{N}, \\ x_{n+1} &= (1 - \lambda_n)x_n + \lambda_n S_{t_n} x_n + \lambda_n\theta_n(y_n - x_n), \end{cases}$$

where $\{e_n\}$ is a bounded error sequence in H. Then $\{x_n\}$ converges strongly to $P_{EP(G)}(u)$.

Proof Note that the family $\{S_t : t > 0\}$ enjoys the property $(\mathscr{A})$ with respect the sequence $\{S_{t_n} : n \in \mathbb{N}\}$ by Lemma 4.16. Moreover, from Lemma 4.13 (3), we get that $F(S_{t_n}) = \bigcap_k F(S_{t_k})$ for any $n \in \mathbb{N}$.
To proceed as in Theorem 4.8, we have only to show that

$$\|S_{t_n}x - S_{t_{n-1}}x\| \leq \frac{|t_n - t_{n-1}|}{t_n}(\|S_{t_n}x\| + \|x\|), x \in H.$$

This last inequality had been implicitly proved in [22, Lemma 5]. □

References

1. Aleyner, A., Reich, S.: An explicit construction of sunny nonexpansive retractions in Banach spaces. Fixed Point Theory Appl. **3**, 295–305 (2005)
2. Allen, G.: Variational inequalities, complementarity problems and duality theorems. J. Math. Anal. Appl. **58**, 1–10 (1977)
3. Aoyama, K., Kimura, Y., Takahashi, W., Toyoda, M.: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350–2360 (2007)
4. Aubin, J.P., Cellina, A.: Differential Inclusions: Set-Valued Maps and Viability Theory. Springer, Berlin (1984)
5. Barbu, V., Precopanu, Th.: Convexity and Optimization in Banach spaces. Editura Academei R.S.R, Bucharest (1978)
6. Benavides, T.D., Acedo, G.L., Xu, H.K.: Construction of sunny nonexpansive retractions in Banach spaces. Bull. Aust. Math. Soc. **66**, 9–16 (2002)
7. Benavides, T.D., Acedo, G.L., Xu, H.K.: Iterative solutions for zeros of accretive operators. Math. Nachr. **248**(249), 62–71 (2003)
8. Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibrium problems. J. Optim. Theory Appl. **90**, 31–43 (1996)
9. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
10. Brezis, H., Nirenberg, L., Stampacchia, G.A.: Remark on Ky Fans minimax principle. Boll. Unione Mat. Ital. **6**, 293–300 (1972)
11. Browder, F.E.: Nonlinear mappings of nonexpansive and accetive type in Banach spaces. Bull. Am. Math. Soc. **73**, 875–882 (1967)
12. Bruck, R.E. Jr.: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251–262 (1973)
13. Bruck, R.E. Jr.: A strongly convergent iterative solution of $0 \in Ux$ for a maximal monotone operator U in Hilbert spaces. J. Math. Anal. Appl. **48**, 114–126 (1974)

14. Byrne, C.: Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Probl. **18**, 441–453 (2002)
15. Byrne, C.L.: A unified treatment of some iterative algorithms insignal processing and image reconstruction. Inverse Probl. **20**, 103–120 (2004)
16. Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in a product space Numer. Algorithms **8**, 221–239 (1994)
17. Censor, Y., Bortfeld, T., Martin, B., Tromo, A.: A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
18. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**, 2071–2084 (2005)
19. Censor, Y., Motova, A., Segal, A.: Perturbed projections and subgradient projections for the multiple-sets split feasibility problem. J. Math. Anal. Appl. **327**, 1244–1256 (2007)
20. Censor, Y., Segal, A.: Iterative projection methods in biomedical inverse problems. In: Censor, Y., Jiang, M., Louis, A.K. (eds.) Mathematical Methods in Biomedical Imaging and Intensity-Modulated Radiation Therapy (IMRT), pp. 65–96. Edizioni della Normale, Pisa (2008)
21. Chidume, C.E., Zegeye, H.: Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps. Proc. Am. Math. Soc. **132**, 831–840 (2003)
22. Colao, V., Lopez Acedo, G., Marino, G.: An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings. Nonlinear Anal. **71**, 2708–2715 (2010)
23. Colao, V., Leustean, L., Lopez, G., Martin-Marquez, V.: Alternative iterative methods for nonexpansive mappings, rates of convergence and applications. J. Convex Anal. **18**(2), 465–487 (2011)
24. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**(1), 117–136 (2005)
25. Ecksteind, J., Bertsekas, D.P.: On the Douglas–Rachford splitting method and the proximal point algorithm for maximal monotone operators. Math. Program. **55**, 293–318 (1992)
26. Eckstein, J.: Approximate iterations in Bregman-function-based proximal algorithms. Math. Program. **83**, 113–123 (1998)
27. Edelstein, M., OBrien, R.C.: Nonexpansive mappings, asymptotic regularity and successive approximations. J. Lond. Math. Soc. **3**, 547–554 (1978)
28. Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequality, vol. III, pp. 103–113. Academic Press, New York (1972)
29. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker Inc., New York (1984)
30. Gol'shtein, E.G., Tret'yakov, N.V.: The gradient method of minimization and -algorithms of convex programming based on Lagrangian functions. Ekonomika i Matematcheskie Metody **11**(4), 730–742 (1975)
31. Güler, O.: On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. **29**, 403–419 (1991)
32. Ha, K.S., Jung, J.S.: Strong convergence theorems for accretive operators in Banach space. J. Math. Anal. Appl. **147**, 330–339 (1990)
33. Halpern, B.: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957–961 (1967)
34. Iusem, A.N., Sosa, W.: Iterative algorithms for equilibrium problems. Optimization **52**(3), 301–316 (2003)
35. Kamimura, S., Takahashi, W.: Approximating solutions of maximal monotone operators in Hilbert space. J. Approx. Theory **106**, 226–240 (2000)
36. Kato, T.: Nonlinear semi-groups and evolution equations. J. Math. Soc. Jpn. **19**, 508–520 (1967)
37. Lim, T.-C.: A fixed point theorem for families of nonexpansive mappings. Pac. J. Math **53**, 487–493 (1974)
38. Leuştean, L.: Rates of asymptotic regularity for Halpern iterations of nonexpansive mappings. In: Calude, C.S., Stefanescu, G., Zimand, M. (eds.) Combinatorics and Related Areas. A Collection of Papers in Honor of the 65th Birthday of Ioan Tomescu, *J. Univers. Comput. Sci.* **13**, 1680–1691 (2007)
39. Mainge, P.E.: Viscosity methods for zeroes of accretive operators. J. Approx. Theory **140**, 127–140 (2006)
40. Marino, G., Colao, V., Muglia, L., Yao, Y.: Krasnoselski-Man iteration for- hierarchical ficed points and equilibrium problem. Bull. Aust. Math. Soc. **79**, 187–200 (2009)
41. Marino, G., Xu, H.K.: Convergence of generalized proximal point algorithm. Comm. Pure Appl. Anal. **3**, 791–808 (2004)
42. Martinet, B.: Regularisation dinequations variationnelles par approximations successives. Rev. Francaise Inform. Recherche Operationnelle **4**, 154–158 (1970)
43. Minty, G.J.: Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29**, 341–346 (1962)
44. Moore, C., Nnoli, B.V.C.: Iterative solution of nonlinear equations involving set-valued uniformly accretive operators. Comput. Math. Appl. **42**, 131–140 (2001)

45. Nakajo, K.: Strong convergence to zeros of accretive operators in Banach spaces. J. Nonlinear Convex Anal. **7**, 71–81 (2006)
46. Oettli, W.: A remark on vector-valued equilibria and generalized monotonicity. Acta Math. Vietnam. **22**, 213–221 (1997)
47. Podilchuk, C.I., Mammone, R.J.: Image recovery by convex projections using a leastsquares constraint. J. Opt. Soc. Am. A **7**, 517–521 (1990)
48. Rockafellar, R.T.: Characterization of the subdifferentials of convex functions. Pac. J. Math. **17**, 497–510 (1966)
49. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
50. Reich, S.: Constructive techniques for accretive and monotone operators. In: Applied Nonlinear Analysis (Proceedings of the third international conference, University of Texas, Arlington, Texas). Academic Press, New York, pp. 335–345 (1979)
51. Sahu, D.R., Wong, N.C., Yao, J.C.: A generalized hybrid steepest-descent method for variational inequalities in Banach spaces. Fixed Point Theory Appl. 1–28. doi: 10.1155/2011/754702 (2011)
52. Sahu, D.R.: Applications of the S-iteration process to constrained minimization problems and split feasibility problems. Fixed Point Theory **12**(1), 187–204 (2011)
53. Sahu, D.R., Yao, J.C.: The prox-Tikhonov regularization method for the proximal point algorithm in Banach spaces. J. Glob. Optim. **51**(4), 641–655 (2011)
54. Takahashi, W., Ueda, Y.: On Reich's strong convergence theorems for resolvents of accretive operators. J. Math. Anal. Appl. **104**, 546–553 (1984)
55. Takahashi, W.: Nonlinear Functional Analysis, Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama (2000)
56. Wong, N.C., Sahu, D.R., Yao, J.C.: Solving variational inequalities involving nonexpansive type mappings. Nonlinear Anal. **69**, 4732–4753 (2008)
57. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240–256 (2002)
58. Xu, H.K.: A variable Krasnoselskii-Mann algorithm and the multiple-set split feasibility problem. Inverse Probl. **22**, 2021–2034 (2006)
59. Xu, H.K.: Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. Inverse Probl. **26** (2010). doi: 10.1088/0266-5611/26/10/105018
60. Youla, D.: Mathematical theory of image restoration by the method of convex projections. In: Stark, H. (ed.) Image Recovery Theory and Applications, pp. 29–77. Academic Press, Orlando (1987)
61. Youla, D.: On deterministic convergence of iterations of relaxed projection operators. J. Vis. Comm. Image Represent. **1**, 12–20 (1990)
62. Zeidler, E.: Nonlinear Functional Analysis and Its Applications, Part II: Monotone Operators. Springer, Berlin (1985)

ON THE CONVERGENCE OF APPROXIMANTS OF PSEUDO-CONTRACTIVE SEMIGROUPS IN BANACH SPACES

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Abstract. The purpose of this paper is to estabilish some results on the convergence of approximated fixed point sequences for uniformly lipschitzian semigroups of pseudo-contractive mappings.

1. Introduction

Let X be a Banach space and let C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be a mapping, we denote by the symbol $F(T) := \{x \in C : Tx = x\}$ the set of fixed points for T and by $k(T)$ we denote, whenever it exists, the Lipschitz constant defined by

$$k(T) := \inf\{k \in [0,\infty) : \|Tx - Ty\| \leq k\|x - y\| \text{for all } x, y \in C\}.$$

We recall that T is called

(1) $L-$lipschitzian if $k(T) = L < \infty$,
(2) nonexpansive if $k(T) = 1$ and
(3) contraction if $k(T) < 1$.

One classical method to approximate fixed points for a nonexpansive mapping T is by passing through fixed points of particular contractive mappings.
More precisely, for a fixed element $u \in C$, define for each $t \in (0, 1)$, a contraction G_t by $G_t x = tu + (1 - t)Tx$ for all $x \in C$. Let x_t be the fixed point of G_t, *i.e.*,

$$x_t = tu + (1 - t)Tx_t. \tag{1.1}$$

Browder [2] proved that x_t strongly converges, as $t \to 0$, to a fixed point of the mapping T, in the setting of Hilbert spaces. Later, Reich [12] extended the result to uniformly smooth Banach spaces. Similarly, many authors have studied the behaviour of the approximants $\{x_\varepsilon\}$ defined by

$$0 = \varepsilon R x_\varepsilon + (1 - \varepsilon)(I - T)x_\varepsilon.$$

for nonexpansive self-mappings T in Banach spaces, where $R = I - A$ and $A : C \to C$ is a contraction mapping. In [6], Gwinner proved strong convergence of inexact approximants $\{\widetilde{y}_n\}$ in a uniformly convex Banach space as follows:

Theorem 1.1. *Let X be a uniformly convex Banach space with a weakly sequentially continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive with $F(T) \neq \emptyset$. Let $R : C \to X$ be a continuous, bounded operator. Suppose R is strongly ϕ-accretive. Let $\{b_n\}$ be a sequence in $(0, 1)$ and let $\{\delta_n\}$ be a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\delta_n}{b_n} = 0$. If the approximate solutions $\widetilde{y}_n \in C$ satisfy*

$$\|b_n R\widetilde{y}_n + (1 - b_n)(I - T)\widetilde{y}_n\| \leq \delta_n \text{ for all } n \in \mathbb{N}, \tag{1.2}$$

then $\{\widetilde{y}_n\}$ converges strongly to an element $y^* \in F(T)$ which uniquely solves the variational inequality:

$$\langle Ry^*, J(y^* - v)\rangle \leq 0 \text{ for all } v \in F(T). \tag{1.3}$$

Let $\mathbb{R}^+$ be the set of nonnegative real numbers and let $\mathscr{F} := \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter family of mappings from C to itself. $\mathscr{F}$ is said to be a *strongly continuous semigroup of mappings* if

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(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
(iii) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

Moreover, $\mathscr{F}$ is said to be an *uniformly continuous semigroup of mappings*, if condition (iii) holds uniformly over any bounded subset of C.

We denote by $F(\mathscr{F})$ the set of all common fixed points of $\mathscr{F}$, i.e., $F(\mathscr{F}) := \bigcap_{t\in\mathbb{R}^+} F(T(t))$.

An interesting problem is to modify Browder's result (1.1) to approximate a common fixed point for a semigroup of nonexpansive mappings. Suzuki [15] proved the following implicit iteration process in a Hilbert space.

Theorem 1.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of nonexpansive mappings from C into itself with $F(\mathscr{F}) \neq \emptyset$. Let $\{b_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ a sequence in $(0,\infty)$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = 0$. Fix $u \in C$ and define a sequence $\{y_n\}$ by*

$$y_n = b_n u + (1-b_n)T(t_n)y_n \text{ for all } n \in \mathbb{N}. \tag{1.4}$$

Then $\{y_n\}$ converges strongly to the element of $F(\mathscr{F})$ nearest to u.

Xu [18] extended Suzuki's result to uniformly convex Banach spaces with weakly sequentially continuous duality mappings and he posed the following question. Can the iteration sequence (1.4) provide the same result in Banach spaces that include the L_p spaces, $1 < p < \infty$?
To give a partial answer to the question, we deal with an important and widely studied generalization of nonexpansive mappings, that is the class of pseudo-contractions. We say that a mapping $T : C \to C$ is said to be

(1) *pseudo-contractive* if for all x, y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$;
(2) *ϕ-strongly pseudo-contractive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all x, y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$;
(3) *generalized Φ-pseudo-contractive* (cf.[17]) if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for all x, y in C, there exists $j(x-y)$ in $J(x-y)$ satisfying $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|)$.

We remark that $R = I - T$ is accretive (resp. ϕ-strongly accretive, uniformly accretive) if T is pseudo-contractive (resp. ϕ-strongly pseudo-contractive, generalized Φ-pseudo-contractive), where I is the identity operator.

Recently, applications of semigroups on the existence of solutions to certain partial differential equations had been explored by Hester and Morales in [7]. They proved that the semigroup result directly implies the existence of a unique global solution to a time evolution equation of the form $u' = Au$, where A is a combination of derivatives.
Our concern now is the following:

Problem 1.3. *Does iteration process (1.4) provide the same result for Lipschitz pseudo-contractive semigroups $\mathscr{F}$ even in uniformly convex spaces?*

In this paper, we prove a version of Theorem 1.1 for a uniformly continuous semigroup of pseudocontractive mappings in a Banach space much more general than uniformly convex spaces. This partially settles the open problem posed by Xu [18] and Problem 1.3.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, X is a real Banach space, C is a nonempty, closed and convex subset of X, X^* is the dual space of X and J is the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) := \{j \in X^* : \langle x, j\rangle = ||x||^2 = ||j||^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single-valued.

Recall that X is said to be *smooth* provided the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $S_X = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable* and it is said to be *uniformly Gâteaux differentiable* if for each $y \in$ S, this limit is attained uniformly for $x \in S$. X is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in X$. Classical examples of uniformly smooth Banach spaces are the L_p spaces, for $1 < p < \infty$ (see e.g., [1, 4]).

Let $\{x_n\}$ be a bounded sequence in X. Consider the functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \ x \in X.$$

The infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$. A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $\mathcal{Z}_a(C, \{x_n\})$.

X is said to satisfy property (I) (cf. [9]) if asymptotic center of every bounded sequence in X with respect to closed convex subsets of X consists of exactly one point.

Uniformly convex spaces are examples of this type Banach spaces (cf. [1, 5]). It is known (cf. Lim [8]) that $\mathcal{Z}_a(C, \{x_n\})$ consists of a single point if X is reflexive and uniformly convex in every direction.

We need the following known fact (cf. Morales [11, Proposition 11]).

Lemma 2.1. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a closed and convex subset of X. Suppose $\{x_n\}$ is a bounded sequence in C and $v \in \mathcal{Z}_a(C, \{x_n\})$. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\limsup_{k\to\infty} \langle u - v, J(x_{n_k} - v)\rangle \leq 0 \textit{ for all } u \in C.$$

A semigroup $\mathscr{F} := \{T(t) : t \in \mathbb{R}^+\}$ of Lipschitzian mappings from C into itself, is said *uniformly Lipschitzian* if there exists a constant $L > 0$ such that $\|T(t)x - T(t)y\| \leq L\|x - y\|$ holds for any $t \in \mathbb{R}^+$ and for any $x, y \in C$.

Let C be a nonempty, closed and convex subset of a smooth Banach space X and D a nonempty subset of C. Given an accretive operator $R : C \to X$, we consider the following variational inequality $VI_D(C, R)$:

$$\text{find } z \in D \text{ such that } \langle Rz, J(z - v)\rangle \leq 0 \text{ for all } v \in D.$$

We denote by $\Omega_D(C, R)$ the set of solutions of variational inequality $VI_D(C, R)$.

Remark 2.2. *If R is uniformly accretive and if $\Omega_D(C, R)$ is nonempty, then this last consists of a unique element.*

Proof. Let $z_1, z_2 \in \Omega_D(C, R)$. Then

$$\langle Rz_1, J(z_1 - z_2)\rangle \leq 0$$

and

$$\langle Rz_2, J(z_2 - z_1)\rangle \leq 0.$$

Summing the two inequalities and by the uniform accretivity of R, we get

$$\Phi(\|z_1 - z_2\|) \leq \langle Rz_1 - Rz_2, J(z_1 - z_2)\rangle \leq 0$$

for some strictly increasing function Φ, with $\Phi(0) = 0$. From this last, it is easily derived that $z_1 = z_2$. □

3. Main results

Firstly, we prove a result on the existence of common fixed points for a semigroup of pseudo-contractions.We assume the existence of an approximated fixed point sequence only for countable many elements of the semigroup.

Lemma 3.1. *Let X be a reflexive Banach space satisfying property (I) and let C be a nonempty closed convex subset of X. Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of continuous pseudo-contractive mappings from C into itself and let $\{t_n\}$ be a sequence in $(0, \infty)$ converging to* 0. *Let $\{y_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|y_n - T(t_m)y_n\| = 0$ for all $m \in \mathbb{N}$ and let y^* be the unique element in $\mathcal{Z}_a(C, \{y_n\})$, then $F(\mathcal{T})$ is nonempty and $y^* \in F(\mathcal{T})$.*

Proof. Fix $m \in \mathbb{N}$. Since $T(t_m)$ is continuous and pseudo-contractive, we derive from [10, Theorem 6] that $g_m := (2I - T(t_m))^{-1}$ is a nonexpansive mapping from C into itself.
Since

$$\begin{aligned}
\limsup_{n\to\infty} \|y_n - g_m(y^*)\| &\leq \limsup_{n\to\infty} \|g_m(y_n) - g_m(y^*)\| \\
&\quad + \limsup_{n\to\infty} \|y_n - g_m(y_n)\| \\
&\leq \limsup_{n\to\infty} \|y_n - y^*\| \\
&\quad + \limsup_{n\to\infty} \|(2I - T(t_m))^{-1}(2y_n - T(t_m)y_n) - (2I - T(t_m))^{-1}(y_n)\| \\
&\leq \limsup_{n\to\infty} \|y_n - y^*\| + \limsup_{n\to\infty} \|y_n - T(t_m)y_n\| \\
&= \limsup_{n\to\infty} \|y_n - y^*\|,
\end{aligned}$$

it follows that $g_m(y^*) \in \mathcal{Z}_a(C, \{y_n\})$ and hence $g_m(y^*) = y^*$ for any $m \in \mathbb{N}$.
As a consequence, $y^* \in \bigcap_{n\in\mathbb{N}} F(t_n)$, where $\{t_n\} \subset (0, \infty)$ converges to 0. Applying [16, Proposition 1], it is easily derived that $y^* \in F(\mathscr{F})$. □

Our second lemma proves the existence of approximating fixed point sequences for a lipschitz semigroup under mild assumptions on the Banach space.

Lemma 3.2. *Let X be a Banach space and let C be a nonempty closed convex subset of X. Let $A : C \to X$ be a bounded mapping (i.e. A maps bounded sets into bounded sets) and let $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly continuous semigroup of uniformly Lipschitz mappings.*
Let $\{b_n\}$ be a sequence in $(0, 1)$ and let $\{t_n\}$ and $\{\delta_n\}$ be two sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = \lim_{n\to\infty} \delta_n/t_n = 0.$$

If $\{y_n\} \subset C$ is a bounded sequence of approximate solutions, i.e. it satisfies

$$\|b_n(I - A)y_n + (1 - b_n)(I - T(t_n))y_n\| \leq \delta_n \text{ for all } n \in \mathbb{N} \tag{3.1}$$

then $\lim_{n\to\infty} \|y_n - T(t_m)y_n\| = 0$ for all $m \in \mathbb{N}$.

Proof. Let $L > 0$ be the Lipschitz constant of the semigroup $\mathscr{F}$ and assume that $\{y_n\}$ is a bounded sequence in C satisfying (3.1).
Without loss of generality, we may assume that $\{b_n\}$ is a sequence in $(0, \delta]$ for some $\delta \in (0, 1)$. Since $\{y_n\}$ and $\{Ay_n\}$ are bounded, there exists a constant $K \geq 0$ such that $\|(I - A)y_n\| \leq K$ for all $n \in \mathbb{N}$.
Note that

$$\begin{aligned}
\|(I - T(t_n))y_n\| &= (1 - b_n)^{-1}\|(1 - b_n)(I - T(t_n))y_n + b_n(I - A)y_n - b_n(I - A)y_n\| \\
&\leq (1 - b_n)^{-1}(\delta_n + b_n\|(I - A)y_n\|) \\
&\leq (1 - \delta)^{-1}(\delta_n + Kb_n).
\end{aligned} \tag{3.2}$$

Let $\tilde{d}$ be the metric on X defined by

$$\tilde{d}(x, y) := \sup_{s\in\mathbb{R}^+} \|T(s)x - T(s)y\|.$$

By standard arguments, it is easily derived that

$$\|x-y\| \le \tilde{d}(x,y) \le L\|x-y\| \text{ for any } x,y \in C, \tag{3.3}$$

and that for any $n \in \mathbb{N}$, $T(t_n)$ is nonexpansive with respect to $\tilde{d}$.
Let $[\cdot]$ be the integer part and fix $m \in \mathbb{N}$. Then, for any $n \ge m$,

$$\begin{aligned} \|y_n - T(t_m)y_n\| &\le \tilde{d}(y_n, T(t_m)y_n) \\ &\le \sum_{i=0}^{[t_m/t_n]-1} \tilde{d}(T(it_n)y_n, T((i+1)t_n)y_n) + \tilde{d}(T([t_m/t_n]t_n)y_n, T(t_m)y_n) \\ &= \sum_{i=0}^{[t_m/t_n]-1} \tilde{d}(T^i(t_n)y_n, T^i(t_n)T(t_n)y_n) \\ &\quad + \tilde{d}(T^{[t_m/t_n]}(t_n)y_n, T^{[t_m/t_n]}(t_n)T(t_m - [t_m/t_n]t_n)y_n) \\ &\le (t_m/t_n)\tilde{d}(y_n, T(t_n)y_n) + \tilde{d}(y_n, T(s_n)y_n), \end{aligned} \tag{3.4}$$

where $s_n := t_m - [t_m/t_n]t_n \ge 0$.
Note that by (3.2) and (3.3), we have

$$\tilde{d}(y_n, T(t_n)y_n) \le L(1-\delta)^{-1}(\delta_n + Kb_n),$$

thus (3.4) becomes

$$\|y_n - T(t_m)y_n\| \le L(t_m(1-\delta)^{-1}(\delta_n/t_n + Kb_n/t_n) + \sup_{y \in \{y_n\}} \|y - T(s_n)y\|). \tag{3.5}$$

Observe that

$$s_n = t_m - [t_m/t_n]t_n \le t_n \to 0$$

and hence

$$\lim_{n\to\infty} \sup_{y \in \{y_n\}} \|y - T(s_n)y\| = 0, \tag{3.6}$$

by the uniform continuity of $\mathscr{F}$.
On the other hand and by hypothesis,

$$\lim_{n\to\infty} t_m(1-\delta)^{-1}(\delta_n/t_n + Kb_n/t_n) = 0,$$

which, together with (3.6) and (3.5), implies $\lim_{n\to\infty} \|y_n - T(t_m)y_n\| = 0$ for any fixed $m \in \mathbb{N}$. □

We now prove our main result.

Theorem 3.3. *Let X be a uniformly smooth Banach space, which satisfies property (I). Let $C \subset X$ be nonempty, closed and convex. Let $A : C \to X$ be a bounded and continuous generalized Φ-pseudo-contractive mapping and $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ a uniformly continuous semigroup of uniformly Lipschitz pseudo-contractive mappings from C into itself. Let $\{b_n\}$ be a sequence in $(0,1)$ and let $\{t_n\}$ and $\{\delta_n\}$ be two sequences in $(0,\infty)$ such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = \lim_{n\to\infty} \delta_n/t_n = \lim_{n\to\infty} \delta_n/b_n = 0. \tag{3.7}$$

If the approximate solutions $y_n \in C$ satisfy (3.1) and $\{y_n\}$ is bounded, then

(a) *$F(\mathscr{F})$ is nonempty,*
(b) *$F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C)$, is nonempty and*
(c) *$\{y_n\}$ converges strongly to the unique element $y^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C)$*

Proof. (a) Assume that the approximate solutions $y_n \in C$ satisfy (3.1) and $\{y_n\}$ is bounded. By Lemma 3.2, we have $y_n - T(t_m)y_n \to 0$ as $n \to \infty$ for all $m \in \mathbb{R}$.
Since X has property (I), it follows from Lemma 3.1 that $F(\mathscr{F}) \cap \mathscr{Z}_a(C, \{y_n\})$ is nonempty and singleton. In particular, $F(\mathscr{F}) \neq \emptyset$.

(b) Let $v \in F(\mathscr{F})$. Set $\beta_n = \langle b_n(I-A)y_n+(1-b_n)(I-T(t_n))y_n, J(y_n-v)\rangle$ and $c_v = \sup_{n\in\mathbb{N}} \|y_n-v\|$. Observe that $\beta_n \le \delta_n c_v$ and

$$\begin{aligned}\langle y_n - T(t_n)y_n, J(y_n-v)\rangle &= \langle y_n - v + T(t_n)v - T(t_n)y_n, J(y_n-v)\rangle \\ &= \|y_n - v\|^2 - \langle T(t_n)y_n - T(t_n)v, J(y_n-v)\rangle \\ &\ge 0 \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Thus,

$$\begin{aligned}\langle (I-A)y_n, J(y_n-v)\rangle &= b_n^{-1}\langle b_n(I-A)y_n + (1-b_n)(I-T(t_n))y_n \\ &\quad -(1-b_n)(I-T(t_n))y_n, J(y_n-v)\rangle \\ &= b_n^{-1}\beta_n - b_n^{-1}(1-b_n)\langle (I-T(t_n))y_n, J(y_n-v)\rangle \\ &\le b_n^{-1}\beta_n \\ &\le b_n^{-1}\delta_n c_v. \end{aligned} \tag{3.8}$$

Let y^* be the unique element of $\mathscr{Z}_a(C,\{y_n\})$, which also lies in $F(\mathscr{F})$. By Lemma 2.1, there exists a subsequence $\{y_{n_k}\}$ such that

$$\limsup_{k\to\infty} \langle Ay^* - y^*, J(y_{n_k} - y^*)\rangle \le 0. \tag{3.9}$$

From (3.8), we have

$$\begin{aligned}\|y_{n_k} - y^*\|^2 &= \langle y_{n_k} - Ay_{n_k} + Ay_{n_k} - Ay^* + Ay^* - y^*, J(y_{n_k} - y^*)\rangle \\ &\le b_n^{-1}\delta_{n_k} c_{y^*} + \|y_{n_k} - y^*\|^2 - \Phi(\|y_{n_k} - y^*\|) + \langle Ay^* - y^*, J(y_{n_k} - y^*)\rangle,\end{aligned}$$

which gives us that

$$\Phi(\|y_{n_k} - y^*\|) \le b_{n_k}^{-1}\delta_{n_k} c_{y^*} + \langle Ay^* - y^*, J(y_{n_k} - y^*)\rangle. \tag{3.10}$$

Together with (3.9), this last implies that $\{y_{n_k}\}$ strongly converges to y^*.
Let $v \in F(\mathscr{F})$ and observe that, by (3.8),

$$\begin{aligned}\langle y^* - Ay^*, J(y^*-v)\rangle &= \langle (I-A)y^*, J(y^*-v)\rangle - \langle (I-A)y^*, J(y_{n_k}-v)\rangle \\ &\quad +\langle (I-A)y^*, J(y_{n_k}-v)\rangle - \langle (I-A)y_{n_k}, J(y_{n_k}-v)\rangle \\ &\quad +\langle (I-A)y_{n_k}, J(y_{n_k}-v) \\ &\le |\langle (I-A)y^*, J(y^*-v)\rangle - \langle (I-A)y^*, J(y_{n_k}-v)\rangle| \\ &\quad +\|(I-A)y_{n_k} - (I-A)y^*\|\|J(y_{n_k}-v)\| + b_{n_k}^{-1}\delta_{n_k} c_v.\end{aligned}$$

Since the duality mapping J is single-valued and norm to weak* uniformly continuous on any bounded subset of a Banach space X with a uniformly Gâteaux differentiable norm and $\{y_{n_k}\}$ converges to y^*, we have

$$\langle y^* - Ay^*, J(y^*-v)\rangle \le 0 \text{ for any } v \in F(\mathscr{F}),$$

i.e. $y^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C) \neq \emptyset$.

(c) Suppose that the sequence $\{y_n\}$ does not converge to y^*. As a consequence, there exists $\varepsilon_0 > 0$ and a subsequence $\{y_{n_m}\}$, such that for any $m \in \mathbb{N}$,

$$\|y_{n_m} - y^*\| \ge \varepsilon_0. \tag{3.11}$$

Let z^* be the unique element in $\mathscr{Z}_a(C,\{y_{n_m}\})$ and note that by Lemma 3.1, z^* also belongs to $F(\mathscr{F})$. By Lemma 2.1 and passing to a further subsequence, if necessary, we can assume that

$$\limsup_{m\to\infty} \langle Az^* - z^*, J(y_{n_m} - z^*)\rangle \le 0.$$

Following the same arguments as in (b), we then derive that $y_{n_m} \to z^*$ and that $z^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C)$. Since $\Omega_{F(\mathscr{F})}(I-A, C)$ is singleton, we obtain that $z^* = y^*$, which contradicts (3.11). Hence $\lim_{n\to\infty} y_n = y^*$. □

By the next proposition, we prove the existence of a sequence satisfying (3.1). Moreover we obtain an answer to the problem posed by Xu in [18].

Proposition 3.4. *Let C be a nonempty closed convex subset of a smooth Banach space X, $A : C \to C$ a continuous generalized Φ-pseudo-contractive mapping and $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ a semigroup of pseudo-contractive mappings from C into itself. Let $\{b_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ a sequence in $(0,\infty)$. For each $n \in \mathbb{N}$, define $G_n : C \to C$ by $G_n z := b_n Az + (1-b_n)T(t_n)z$, $y \in C$. Then, there exists exactly one fixed point z_n in C of G_n defined by*

$$z_n = b_n Az_n + (1-b_n)T(t_n)z_n \text{ for all } n \in \mathbb{N}. \tag{3.12}$$

Proof. Set $\Phi_n(\cdot) := b_n\Phi(\cdot)$ for each $n \in \mathbb{N}$. Then the mapping $G_n : C \to C$ is continuous and generalized Φ_n-pseudo-contractive. Indeed, for x, y in C, we have

$$\begin{aligned}\langle G_n x - G_n y, J(x-y)\rangle &= b_n\langle Ax - Ay, J(x-y)\rangle + (1-b_n)\langle T(t_n)x - T(t_n)y, J(x-y)\rangle \\ &\le b_n(\|x-y\|^2 - \Phi(\|x-y\|)) + (1-b_n)\|x-y\|^2 \\ &= \|x-y\|^2 - \Phi_n(\|x-y\|).\end{aligned}$$

Note also that $\Phi_n(\cdot)$ is a strictly increasing function with $\Phi_n(0) = 0$. By Xiang [17, Theorem 2.1], G_n has a unique fixed point z_n in C. □

Corollary 3.5. *Let X be a uniformly smooth Banach space, which satisfies property (I). Let $C \subset X$ be nonempty, closed and convex. Let $A : C \to X$ be a bounded and continuous generalized Φ-pseudo-contractive mapping and $\mathscr{F} = \{T(t) : t \in \mathbb{R}^+\}$ a uniformly continuous semigroup of uniformly Lipschitz pseudo-contractive mappings from C into itself. Let $\{b_n\}$ be a sequence in $(0,1)$, let $\{t_n\}$ and $\{\delta_n\}$ be two sequences in $(0,\infty)$ such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} b_n/t_n = \lim_{n\to\infty} \delta_n/t_n = \lim_{n\to\infty} \delta_n/b_n = 0 \tag{3.13}$$

and let $\{z_n\}$ be defined by (3.12).
If $\{z_n\}$ is bounded then

(a) *$F(\mathscr{F})$ is nonempty,*
(b) *$F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C)$, is nonempty and*
(c) *$\{z_n\}$ converges strongly to the unique element $y^* \in F(\mathscr{F}) \cap \Omega_{F(\mathscr{F})}(I-A, C)$*

Remark 3.6. *We remark that,*

(a) *in both Lemma 3.2 and Theorem 3.3, if the sequence $\{t_n\}$ can be chosen so that, for $n \ge m$, $t_m/t_n \in \mathbb{N}$ (e.g. $t_n = a^{-n}$ for some $a \in \mathbb{N}$), the uniform continuity condition on the semigroup can be weakened by only assuming strong continuity;*
(b) *in Theorem 3.3, we prove the existence of a solution of a variational inequality problem on the set $F(\mathscr{F})$, which can fail to be convex.*
(c) *the asymptotic center technique is used in Theorem 3.3. Therefore, our approach is different from the results recently studied in Sahu, Wong and Yao [13].*

References

[1] R. P. Agarwal, Donal O'Regan and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Series: Topological Fixed Point Theory and Its Applications, 6, Springer New York, 2009.
[2] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Archive for Rational Mechanics and Analysis, **24** (1967), 82–90.
[3] C.E. Chidume and H. Zegeye, *Strong convergence theorems for common fixed points of uni- formly L-Lipschitzian pseudocontractive semigroups*, Appl. Anal., **86** (2007), 353–366.
[4] L. Cioranescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Kluwer Aacademic Publishers, Dordrecht, 1990.
[5] M. Edelstein, Fixed point theorems in uniformly convex Banach spaces, Proc. Amer. Math. Soc., **44** (1974), 369-374.
[6] J. Gwinner, *On the Convergence of Some Iteration Processes in Uniformly Convex Banach Spaces*, Proc. Amer. Math. Soc., 71 (1978), 29–35.
[7] A. Hester and C. H. Morales, *Semigroups generated by pseudo-contractive mappings under the Nagumo condition*, J. Diff. Eq., **245** (2008), 994–1013.
[8] T. C. Lim, *On asymptotic centres and fixed points for nonexpansive mappings*, Canad. J. Math., **32** (1980), 421–430.
[9] G. Marino and H. K. Xu, *Asymptotic centers, inward sets and fixed points*, Comm. Pure Appl. Anal., **10**(2003),55–63.

[10] R. H. Martin, *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., **179** (1973), 399-414.
[11] C. H. Morales, *Variational inequalities for Φ-pseudo-contractive mappings*, Nonlinear Analysis, **75** (2012), 477–484.
[12] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75** (1980), 287-292.
[13] D. R. Sahu, N. C. Wong, and J.C. Yao, *A unified hybrid iterative method for solving variational inequalities involving generalized pseudo-contractive mappings*, SIAM J. Control Opt., 2012, in press.
[14] D. R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Comment. Math. Univ. Carolin., **46** (2005), 653–666.
[15] T. Suzuki, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces*, Proc. Amer. Math. Soc., **131** (2002), 2133–2136.
[16] T. Suzuki, *The set of common fixed points of a one-parameter continuous semigroup of mappings is $F(T(1)) \cap F(T(\sqrt{2}))$* Proc. Amer. Math. Soc., **134** (2006), 673–681
[17] Chang He Xiang, *Fixed point theorem for generalized Φ-pseudo-contractive mappings*, Nonlinear Anal., **70** (2009), 277–279.
[18] H. K. Xu, *A strong convergence theorem for contraction semigroups in Banach spaces*, Bull. Austral. Math. Soc., **72** (2005), 371–379.
[19] N. C. Wong, D. R. Sahu, and J.C. Yao, *Solving variational inequalities involving nonexpansive type mappings*, Nonlinear Anal., **69** (2008) 4732–4753.

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Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay

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Abstract We establish existence of solutions for a second-order differential equation with non-instantaneous impulses and delay on an unbounded interval. A compactness criterion in a certain class of functions is established, which then permits to reduce the differential equation to an equivalent fixed-point problem.

Keywords Impulsive differential equation · Fixed-point theorem · Compactness · Unbounded interval

Mathematics Subject Classification 34K45 · 46B50 · 47D09 · 47H10

1 Introduction

We are concerned with the existence of solutions for a differential equation with non-instantaneous impulses and delay of the form

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$$(P) \qquad \begin{cases} x''(t) = Ax(t) + f(t, x(t), x(\sigma(t))), & \text{a.e.} t \in (0, t_1] \cup \bigcup_{i=1}^{N} (s_i, t_{i+1}] \\ x(t) = \gamma_i(t, x(t)), & t \in \bigcup_{i=1}^{N} (t_i, s_i], \\ x(t) = \phi(t), & t \in [-r, 0], \quad x'(0) = \phi'(0) = \eta, \end{cases}$$

where x maps $[-r, +\infty)$ into $(\mathbb{R}^n, |\cdot|)$ ($|\cdot|$ being a norm, but not necessarily the Euclidean norm), $\mathrm{T} := \{0 < t_1 < \cdots < t_N\} \subset [0, +\infty)$, $t_{N+1} := +\infty$, $s_i \in (t_i, t_{i+1})$ for each $i = 1, \ldots, N$ and A is a real $n \times n$ matrix.

Impulsive differential equations have been widely investigated (see [11,13,14,16,18,20,22,23] and references therein), but only the instantaneous case has been deeply studied. On the other hand, in many real-world applications, the reaction of a system is transitory but lasts for a finite time interval.

To give a concrete example in Hernandez and O'Regan [13], the following simplified situation concerning the hemodynamical equilibrium has been pointed out. In the case of a hyperglycemic patient, an intravenous drug can be prescribed (insulin). The introduction of the drug into the bloodstream causes an abrupt change in the system, followed by a continuous process until the drug is completely absorbed.

We model the situation by considering a non-instantaneous impulse which starts with a jump and continuously proceeds for a finite time.

On the other hand, delay differential equations have been deeply studied and incorporated into models in different branches of science (see, for example, [4,5,25] and the book [7]).

Focusing our attention to pharmacokinetics, we observe that in Perelson et al [19], delay equations have been applied to the study of the correlation between the administration of drugs and the decline of the viral load in HIV infections.

Our approach consists in translating the problem (P) into a fixed point problem in the Banach space

$$\begin{aligned} \mathrm{BPC}_{\mathrm{T}}[-r, +\infty) := \{ & y : [-r, +\infty) \setminus \mathrm{T} \to \mathbb{R}^n \,|\, y \text{ is bounded and continuous in } t \notin \mathrm{T}, \\ & \text{there exist } y(t_k^-) = y(t_k) \text{ and } y(t_k^+) < \infty,\ t_k \in \mathrm{T}\}, \end{aligned}$$

where $y(t_k^+)$ and $y(t_k^-)$ represent the right limit and the left limit at t_k, respectively. This space is endowed with the supremum norm

$$\|y\|_\infty := \sup\{|y(t)| : \ t \in [-r, +\infty)\}.$$

We shall introduce two operators, one concerning the differential equation and the other related to the non-instantaneous impulses, in order to equivalently convert the solution of the problem (P) to a fixed point of the sum of the above-mentioned operators.

To this end, we use a generalization, given in [9], of the following well-known theorem.

Theorem 1.1 (Krasnosel'skii [15]) *Let C be a closed, convex and bounded subset of a Banach space. Let $T = A + B$, where A is a contraction, B is completely continuous and $T(C) \subset C$. Then, T has a fixed point in C.*

To the best of our knowledge, this approach represents a new strategy for differential equations with non-instantaneous impulses.

Finally, we remark that we will prove the existence of strong solutions defined on an unbounded interval, i.e., the existence of a function $x \in \mathrm{BPC}_{\mathrm{T}}[-r, +\infty)$ twice differentiable for any $t \notin \mathrm{T}$ and which satisfies (P). In this direction, our result represents a step forward in the study of this class of equations.

The rest of the paper is structured as follows. In the next section, notations are introduced together with preliminary results. Incidentally, a new compactness criterion for piecewise continuous functions is obtained. In Sect. 3, problem (P) is translated into a fixed-point problem, by introducing a compact integral operator. In Sect. 4, our main result is proved and the existence of solutions is obtained (Theorem 4.1 and Corollary 4.2).

2 Notations and preliminaries

In the sequel, we indicate by $|x|$ the norm of a vector $x \in \mathbb{R}^n$ and by $|||A||| := \sup\{|Ax| : |x| = 1\}$ the norm of a linear operator A in $\mathbb{R}^n$ (i.e., an $n \times n$ matrix). If Q is a topological space, we denote by $\mathrm{BC}(Q)$ the space of continuous and bounded functions defined in Q into $\mathbb{R}^n$ endowed with the supremum norm

$$\|u\|_\infty := \sup_{t \in Q} |u(t)|,$$

whenever $u \in \mathrm{BC}(Q)$.

Let $r > 0$ and let $\Theta := \{0 < d_1 < \cdots < d_M\}$ be a finite subset of $[-r, +\infty)$. By $\mathrm{BPC}_\Theta[-r, +\infty)$, we denote the Banach space of bounded and continuous functions $y : [-r, +\infty) \setminus \Theta \to \mathbb{R}^n$ such that the limits

$$y(d_k^-) := \lim_{t \to d_k^-} y(t) = y(d_k) \;\text{ and }\; y(d_k^+) := \lim_{t \to d_k^+} y(t) < +\infty,$$

exist for any $k = 1, \ldots, M$.

Let E be a Banach space and let C be a subset of E. We also cite the following well-known definition.

Definition 2.1 [1] The Hausdorff measure of noncompactness of a bounded set C is defined as

$$\chi(C) := \inf\{\varepsilon > 0 : \text{ there exists a finite } \varepsilon\text{-net for } C\}.$$

We recall the following properties of χ :

1. $\chi(C) = 0$ if and only if $\overline{C}$ is compact,
2. $\chi(\overline{\mathrm{co}}\, C) = \chi(C)$,
3. $\chi([0, 1] \cdot C) = \chi(C)$,
4. $\chi(D) \leq \chi(C)$ for $D \subset C$,
5. $\chi(\lambda C) = |\lambda| \chi(C)$,
6. $\chi(C + D) \leq \chi(C) + \chi(D)$,
7. $\chi(C \cup D) = \max\{\chi(C), \chi(D)\}$,

whenever C and D are bounded subsets of E and λ is a real number.

Definition 2.2 A map $F : C \to E$ is said to be completely continuous if it is continuous and maps bounded subsets of C to relatively compact sets.

2.1 Compactness criteria in $\mathrm{BPC}_\Theta[-r, +\infty)$

Definition 2.3 A subset Ω in $\mathrm{BPC}_\Theta[-r, +\infty)$ is quasi-equicontinuous if, for every $u \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|u(\tau_1) - u(\tau_2)| < \varepsilon$ whenever $|\tau_1 - \tau_2| < \delta$ and $\tau_1, \tau_2 \in [-r, d_1]$, $\tau_1, \tau_2 \in (d_k, d_{k+1}]$ for some $k = 1, \ldots, M$ or $\tau_1, \tau_2 \in (d_M, +\infty)$.

For a closed and bounded domain, the following is well known.

Lemma 2.4 *A set $\Omega \subset PC[a,b]$ is relatively compact in $PC[a,b]$ if and only if Ω is bounded and quasi-equicontinuous.*

For unbounded domains, the above conditions do not guarantee the relative compactness, as the next example shows.

Example 2.5 Let $\{u_n\} \subset \mathrm{BC}(\mathbb{R})$ be the sequence defined by $u_n(t) = \arctan(t+n)$. Then, it can be easily seen that the set $\{u_n\}$ is bounded and equicontinuous, but it is not relatively compact.

For the space of continuous and bounded functions on a general topological space Q, we point out the following compactness criteria due to Bartle.

Theorem 2.6 [2] *Let $\Omega \subset BC(Q)$ be a bounded subset. The following is equivalent:*

1. *Ω is relatively compact.*
2. *For any positive ϵ, there is a partition $A_1, \ldots, A_n$ of Q such that if s, t belong to the same set A_i then*

$$|f(t) - f(s)| < \epsilon$$

for every $f \in \Omega$.

To the best of our knowledge, the next lemma represents a new result.

Lemma 2.7 *A bounded set Ω of $BPC_\Theta[-r,+\infty)$ is relatively compact if and only if it is quasi-equicontinuous and has the property that for any $\varepsilon > 0$, there exists $L = L(\varepsilon) > d_M$ such that*

$$\chi(\Omega_{|[L,+\infty)}) < \varepsilon. \tag{2.1}$$

Proof It is easy to see that if Ω is relatively compact, and then, it is quasi-equicontinuous and $\chi(\Omega_{|[L,+\infty)}) = 0$ for any L.

To prove the converse, fix $\varepsilon > 0$ and note that (2.1) implies that there exists $L > d_M$ and a finite $\varepsilon/3-$net $\{\varphi_i : i = 1\ldots, m\}$ for $\Omega_{|[L,+\infty)}$. Since the set $\{\varphi_1, \ldots, \varphi_m\}$ is compact in $\mathrm{BC}[L,+\infty)$, by Theorem 2.6, there exists a partition $\{V_1, \ldots, V_p\}$ of subsets of $[L,+\infty)$ with the property that

$$|\varphi_i(t) - \varphi_i(s)| < \varepsilon/3, \tag{2.2}$$

whenever $i \in \{1, \ldots, m\}$ and $t, s \in V_j$, for some index j.

For $j \in \{1, \dots, p\}$, let $t, s \in V_j$ and $v \in \Omega$. Since $\{\varphi_i : i = 1 \dots, m\}$ is a $\varepsilon/3-$net for Ω, there exists $\hat{i} \in \{1, \dots, m\}$ such that

$$\|u - \varphi_{\hat{i}}\|_\infty < \varepsilon/3.$$

Combining this last with (2.2), we obtain

$$\begin{aligned} |u(t) - u(s)| &= |(u(t) - \varphi_{\hat{i}}(t)) + (\varphi_{\hat{i}}(t) - \varphi_{\hat{i}}(s)) + (\varphi_{\hat{i}}(s) - u(s))| \\ &\le 2\|u - \varphi_{\hat{i}}\|_\infty + |\varphi_{\hat{i}}(t) - \varphi_{\hat{i}}(s)| \le \varepsilon, \end{aligned} \tag{2.3}$$

uniformly on $u \in \Omega$. Let $Q := [-r, d_1) \cup \bigcup_{k=1}^{M-1}(d_k, d_{k+1}) \cup (d_M, L)$. By the quasi-equicontinuity of Ω, it follows that there exists a finite partition $U_1, \dots, U_l$ with the property that

$$|u(t) - u(s)| < \varepsilon,$$

whenever $u \in \Omega$ and $t, s \in U_j$ for some $j \in \{1, \dots, l\}$.

It is a consequence of (2.3) that, for any $\varepsilon > 0$, there exists a finite partition $\Xi := \{U_1, \dots, U_l, V_1, \dots, V_p\}$ of Q with the property that $|u(t) - u(s)| < \varepsilon$, whenever $u \in \Omega$ and t and s belong to the same element of Ξ. By Theorem 2.6 and since $\mathrm{BPC}_\Theta[-r, +\infty)$ is a closed subspace of $\mathrm{BC}(Q)$, it turns that Ω is relatively compact in $\mathrm{BPC}_\Theta[-r, +\infty)$. □

Corollary 2.8 *Let $F : BPC_\Theta[-r, +\infty) \to BPC_\Theta[-r, +\infty)$ be an operator. Suppose that for any bounded set $\Omega \subset BPC_\Theta[-r, +\infty)$, $F(\Omega)$ is a bounded and quasi-equicontinuous subset of $BPC_\Theta[-r, +\infty)$. Suppose also that for any $\varepsilon > 0$, there exists $L > d_M$ such that*

$$\chi(F(\Omega)_{|[L,+\infty)}) < \varepsilon. \tag{2.4}$$

Then, F is a compact operator.

2.2 Cosine family of bounded linear mappings

Firstly, we recall definitions, notations and useful facts regarding the *cosine families* (see [8,10,24] for more details).

Definition 2.9 A one-parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear mappings on $\mathbb{R}^n$ into itself is called *a strongly continuous cosine family* if and only if:

1. $C(t + s) + C(t - s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$;
2. $C(0) = I$, where I is the identity map;
3. $C(\cdot)x \in C(\mathbb{R}, \mathbb{R}^n)$, for all $x \in \mathbb{R}^n$.

The *sine family* $(S(t))_{t\in\mathbb{R}}$ is defined as

$$S(t)x := \int_0^t C(s)x\mathrm{d}s.$$

The relations below follow immediately from the above definition:

(u1) $S(t)x$ is continuous in $t \in \mathbb{R}$, $S(0) = 0$ and $S(-t) = -S(t)$, for all $t \in \mathbb{R}$;
(u2) $C(t) = C(-t)$, for all $t \in \mathbb{R}$;
(u3) $C(t)S(s) = S(s)C(t)$, $C(t)C(s) = C(s)C(t)$ and $S(t)S(s) = S(s)S(t)$ for all $t, s \in \mathbb{R}$;
(u4) $S(t + s) + S(t - s) = 2S(t)C(s)$, for all $t, s \in \mathbb{R}$;
(u5) $S(t + s) = S(t)C(s) + C(t)S(s)$, for all $t, s \in \mathbb{R}$.

Definition 2.10 [24] The infinitesimal generator of a strongly continuous cosine family is the operator $A : \mathrm{Dom}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$A := \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)\Big|_{t=0}$$

and $\mathrm{Dom}(A) := \{x \in \mathbb{R}^n : C(t)x \text{ is continuously differentiable at } t = 0\}$.

Note that, for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)x = \mathrm{AC}(t)x = C(t)Ax$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} C(t)x = \int_0^t AC(s)x\mathrm{d}s = A \int_0^t C(s)x\mathrm{d}s = AS(t)x. \tag{2.5}$$

Then, it is easily derived from $(u5)$ that

(u6) $C(t+s) = C(t)C(s) + AS(t)S(s)$, for all $t, s \in \mathbb{R}$.

Definition 2.11 [24] A cosine family $(C(t))_{t\in\mathbb{R}}$ (resp. a sine family $(S(t))_{t\in\mathbb{R}}$) is uniformly bounded if there exists $M_C > 0$ (resp. $M_S > 0$) such that

$$|||C(t)||| \le M_C \text{ (resp., } |||S(t)||| \le M_S), \quad t \in \mathbb{R}.$$

If $M_C = 1$ (resp., $M_S = 1$), we say that the family is 1-uniformly bounded.

The above definition is illustrated in the following examples.

Example 2.12 Let us consider $\mathbb{R}^n$ with the maximum norm. Let A be a bounded and linear operator in $\mathbb{R}^n$ such that $A = \left(\frac{-a_{i,j}^2}{n}\right)_{i,j\in\{1,\ldots,n\}}$ with $a_{i,j} \ge 1$. Then,

$$C(t) = \frac{1}{n}\begin{pmatrix} \cos(a_{1,1}t) & \cos(a_{1,2}t) & \ldots & \cos(a_{1,n}t) \\ \cos(a_{2,1}t) & \cos(a_{2,2}t) & \ldots & \cos(a_{2,n}t) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(a_{n,1}t) & \cos(a_{n,2}t) & \ldots & \cos(a_{n,n}t) \end{pmatrix}$$

is a 1-uniformly bounded cosine family generated by

$$A = \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)\Big|_{t=0}.$$

Indeed,

$$|C(t)x| = \frac{1}{n} \max_{1\le i\le n} |\sum_{j=1}^n \cos(a_{i,j}t)x_j| \le \frac{1}{n}\sum_{j=1}^n |x_j| \le |x|.$$

The sine family is given by

$$S(t) = \frac{1}{n}\left(\frac{1}{a_{i,j}} \sin(a_{i,j}t)\right)_{i,j\in\{1,\ldots,n\}}.$$

Again $S(t)$ is 1-uniformly bounded.

We also give an example in an infinite-dimensional Banach space.

Example 2.13 [8,12] Let $E = L^2[0, \pi]$ and define A by $Af(\xi) := f''(\xi)$ with $D(A) = \{f \in H^2(0, \pi) : f(0) = 0,\ f(\pi) = 0\}$. The operator A is the generator of a cosine family on E defined by

$$C(t)f = \sum_{n=1}^{\infty} \cos(nt)\langle f, z_n\rangle z_n$$

where $z_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$. The sine family is given by

$$S(t)f = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle f, z_n\rangle z_n.$$

Moreover, $|||C(t)||| = |||S(t)||| = 1$.

Remark 2.14 We stress that the case depicted in Example 2.12 can be generalized to obtain sufficient conditions for an operator A to generate 1-uniformly bounded families of cosine and sine. Indeed, let A be a self-adjoint operator on a Hilbert space. Assume that A satisfies the coercivity condition

$$\langle Ax, x\rangle \le -\|x\|^2 \qquad \forall x \in D(A).$$

Then, A is the infinitesimal generator of the cosine family $C(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} A^n$. Moreover, $|||C(t)||| \le 1$ and $|||S(t)||| \le 1$. (See [8, pp. 145–147] and [24]).

From now on, we shall assume that the cosine and sine families are 1-uniformly bounded.

We note that the uniform boundedness of cosine and sine families guarantees their uniform continuity in t, as the following remark explains.

Remark 2.15 The sine family satisfies the property

$$|(S(\tau_1) - S(\tau_2))x| \le \int_{\tau_1}^{\tau_2} |C(s)x|\mathrm{d}s \le |\tau_1 - \tau_2||x|, \quad \tau_1, \tau_2 \in \mathbb{R}.$$

The uniform continuity of $S(t)x$, therefore, follows for all fixed x. As a rule, $S(t)$ is uniformly continuous. Furthermore, from (2.5),

$$|(C(\tau_1) - C(\tau_2))x| \le \int_{\tau_1}^{\tau_2} |AS(s)x|\mathrm{d}s \le |||A||||\tau_1 - \tau_2||x|$$

and the uniform continuity of $C(t)$ follows.

2.3 A fixed-point theorem

In order to prove our results, we will use a Krasnoselskii's type fixed-point theorem proved by Garcia-Falset [9].

Definition 2.16 [3,9,21] We say that $S : C \to E$ is a ρ-contraction if there exists a continuous nondecreasing function $\rho : [0, +\infty) \to [0, +\infty)$ and $\rho(t) < t$ for which the inequality $\|S(x) - S(y)\| \le \rho(\|x - y\|)$ holds for any $x, y \in C$.

We note that each ρ-contraction is continuous and has at most one fixed point. Moreover, ρ-contractions are a proper generalization of classical contractions. In fact, a classical k-contraction is a ρ-contraction with $\rho(t) = kt$, but the converse does not hold as the following example shows.

Example 2.17 [17] Let $X = C(\mathbb{R}, [0, 1])$ and $S : X \to X$ defined as

$$(S(x))(t) = x(t) - \frac{x^4(t) - \sin^2(t)}{4}.$$

[17, Conclusion 4.2] ensures that, if $\rho(t) = t\left(1 - \frac{t^3}{8}\right)$, then

$$\|S(x) - S(y)\|_\infty \le \rho(\|x - y\|_\infty).$$

However, S fails to be a contraction.

Theorem 2.18 [9,21] *Let E be a Banach space. Let $F, S : E \to E$ be mappings satisfying:*

(i) *S is a ρ-contraction,*
(ii) *F is completely continuous.*

Let

$$\zeta(F + S) := \{x \in E : x = \lambda S(\frac{x}{\lambda}) + \lambda F x,\ 0 < \lambda < 1\}.$$

Then, either $\zeta(F + S)$ is unbounded or $F + S$ has a fixed point.

The following corollary easily follows from the above theorem.

Corollary 2.19 *Let T_1 and T_2 be finite (and disjoint) subsets of $[-r, +\infty)$. Suppose*

1. *$T : BPC_{\mathrm{T}_1 \cup \mathrm{T}_2}[-r, +\infty) \to BPC_{\mathrm{T}_1 \cup \mathrm{T}_2}[-r, +\infty)$ is completely continuous,*
2. *$\Gamma : BPC_{\mathrm{T}_1 \cup \mathrm{T}_2}[-r, +\infty) \to BPC_{\mathrm{T}_1 \cup \mathrm{T}_2}[-r, +\infty)$ is a ρ-contraction,*
3. *$\mathfrak{T} : BPC_{\mathrm{T}_1 \cup \mathrm{T}_2}[-r, +\infty) \to BPC_{\mathrm{T}_1}[-r, +\infty)$ (i.e., $\mathfrak{T}x$ is continuous on T_2) such that*

$$\mathfrak{T}x = Tx + \Gamma x.$$

If the set

$$\zeta(T + \Gamma) := \{x \in BPC_{\mathrm{T}_1 \cup \mathrm{T}_2} : x = \lambda \Gamma\left(\frac{x}{\lambda}\right) + \lambda T x \text{ for } 0 < \lambda < 1\}$$

is bounded, then $\mathfrak{T}$ admits fixed points in $BPC_{\mathrm{T}_1}[-r, +\infty)$.

3 The integral problem

From now on, we assume that the functions $f, \phi, \sigma, \gamma_i$ and the matrix A satisfy the following properties:

(h_A) The linear operator $A : \mathrm{Dom}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ is the infinitesimal generator of a strongly continuous, 1-uniformly bounded cosine family $(C(t))_{t \in \mathbb{R}}$. Assume, in addition, that the corresponding sine family $(S(t))_{t \in \mathbb{R}}$ is 1-uniformly bounded, as well.
(h_f) $f : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is such that

- $f(t, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for a.e. fixed $t \in [0, +\infty)$;
- there exist a Lebesgue integrable function $p : [0, +\infty) \to [0, +\infty)$ and a continuous nondecreasing function $\Psi : [0, +\infty) \to [1, +\infty)$ for which

$$|f(t, x, y)| \le p(t)\Psi(|x| + |y|), \text{ for a.e. } t \ge 0,\ x, y \in \mathbb{R}^n,$$

and

$$\int_0^\infty \frac{\mathrm{d}s}{\Psi(s)} = +\infty. \tag{3.1}$$

(h_σ) $\sigma : [0,+\infty) \to [-r,+\infty)$ is a continuous and increasing function, such that $\sigma(t) \le t$ for any $t \in [0,+\infty)$.
$(h_{\phi,\gamma})$ The function ϕ belongs to $C^1([-r,0],\mathbb{R}^n)$, and for any $i = 1,\dots,N$, $\gamma_i : (t_i,s_i] \times \mathbb{R}^n \to \mathbb{R}^n$ is a ρ-contraction with respect to the second variable, such that there exists $\gamma_i(t_i^+,x)$ for all $x \in \mathbb{R}^n$ and

$$|\gamma_i(t,x)| \le a_i(t) + b_i|x|$$

where $a_i(t)$ is a bounded function and $b_i \in (0,1)$.

Let $\Sigma := \{s_1,\dots,s_N\}$. In order to apply Theorem 2.18 and Corollary 2.19 we define, for $x \in \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r,+\infty)$,

$$T_0x(t) := \begin{cases} \phi(t), & t \in [-r,0] \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s, & t \in (0,t_1] \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

For all $i = 1,\dots,N$, taking into account the assumption $t_{N+1} := +\infty$, we define

$$T_ix(t) := \begin{cases} C(t-s_i)\gamma_i(s_i,x(s_i)) + \int_{s_i}^t S(t-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s, & t \in (s_i,t_{i+1}] \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

and

$$\Gamma x(t) := \begin{cases} \gamma_i(t,x(t)), & t \in (t_i,s_i] \quad \text{and} \quad i = 1,\dots,N \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

In this way, we can write

$$Tx(t) := \sum_{i=0}^N T_ix(t) \text{ and } \mathfrak{T}x(t) := Tx(t) + \Gamma x(t)$$

or equivalently

$$\mathfrak{T}x(t) := \begin{cases} \phi(t), & t \in [-r,0] \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s, & t \in (0,t_1] \\ \gamma_i(t,x(t)), & t \in (t_i,s_i] \\ & i = 1,\dots,N \\ C(t-s_i)\gamma_i(s_i,x(s_i)) + \int_{s_i}^t S(t-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s, & t \in (s_i,t_{i+1}]. \\ & i = 1,\dots,N \end{cases} \tag{3.5}$$

In the next propositions, we will prove some properties of the operator $\mathfrak{T}$ which will be useful in the sequel.

Proposition 3.1 *The operator* $\mathfrak{T}$ *maps* $BPC_{\mathrm{T}\cup\Sigma}[-r,+\infty)$ *into* $BPC_{\mathrm{T}}[-r,+\infty)$*. Moreover, for any fixed* $x \in BPC_{\mathrm{T}}[-r,+\infty)$*,* $\mathfrak{T}x$ *is twice differentiable in* $t \in (0,t_1] \cup \bigcup_{i=1}^{N}(s_i,t_{i+1}]$*. In addition, the fixed points of* $\mathfrak{T}$ *are solutions of the problem* (P)*.*

Proof At first, we show that $\mathfrak{T}$ maps $\mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r,+\infty)$ into $\mathrm{BPC}_{\mathrm{T}}[-r,+\infty)$, that is, for any fixed $x \in \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r,+\infty)$, $\mathfrak{T}x$ is continuous at each $(s_i)_{i=1,\ldots,N}$ and right-continuous at each $(t_i)_{i=1,\ldots,N}$.

Indeed, for any fixed $i = 1,\ldots,N$, we have

$$\begin{aligned}\lim_{t\to s_i^+}\mathfrak{T}x(t) &= \lim_{t\to s_i^+} C(t-s_i)\gamma_i(s_i,x(s_i)) + \int_{s_i}^{t} S(t-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s\\ &= \gamma_i(s_i,x(s_i)),\end{aligned}$$

since $C(t)$ is strongly continuous and $C(0)=I$. On the other hand,

$$\lim_{t\to s_i^-}\mathfrak{T}x(t) = \gamma_i(s_i,x(s_i)) = \mathfrak{T}x(s_i).$$

Moreover, by assumption $(h_{\phi,\gamma})$ and for any fixed $k=1,\ldots,N$, there exists

$$\lim_{t\to t_k^+}\mathfrak{T}x(t) = \lim_{t\to t_k^+}\Gamma x(t) = \lim_{t\to t_k^+}\gamma_k(t,x(t)).$$

Now, we prove that $\mathfrak{T}x$ is a bounded function. To this end, we note that by $(h_{\phi,\gamma})$, (h_f) and the 1-uniform boundedness of $C(\cdot)$ and $S(\cdot)$, for $t \in [-r,t_1]$:

$$|\mathfrak{T}x(t)| = |T_0x(t)| \le \|\phi\|_\infty + |\eta| + \Psi(2\|x\|_\infty)\int_0^{t_1} p(s)\mathrm{d}s. \tag{3.6}$$

In a similar way, for fixed $i \in \{1,\ldots,N\}$ and $t \in [s_i,t_{i+1}]$, we get

$$|\mathfrak{T}x(t)| = |T_ix(t)| \le \|a_i\|_\infty + b_i\|x\|_\infty + \Psi(2\|x\|_\infty)\int_{s_i}^{t_{i+1}} p(s)\mathrm{d}s. \tag{3.7}$$

On the other hand, $(h_{\phi,\gamma})$ guarantees that for $t \in \bigcup_{i=1}^{N}(t_i,s_i]$

$$|\Gamma x(t)| \le \max\{\|a_i\|_\infty + b_i\|x\|_\infty : i=1,\ldots,N\}. \tag{3.8}$$

By (3.6), (3.7), (3.8), we conclude that for all $t \in [-r,+\infty)$

$$\begin{aligned}|(\mathfrak{T}x)(t)| &\le |T_0x(t)| + \sum_{i=1}^{N}|T_ix(t)| + |\Gamma x(t)|\\ &\le \|\phi\|_\infty + |\eta| + \Psi(2\|x\|_\infty)\int_0^{t_1} p(s)\mathrm{d}s\\ &\quad + \sum_{i=1}^{N}\left(\|a_i\|_\infty + b_i\|x\|_\infty + \Psi(2\|x\|_\infty)\int_{s_i}^{t_{i+1}} p(s)\mathrm{d}s\right)\\ &\quad + \max\{\|a_i\|_\infty + b_i\|x\|_\infty : i=1,\ldots,N\}\\ &\le \|\phi\|_\infty + |\eta| + \Psi(2\|x\|_\infty)\|p\|_1\\ &\quad + (N+1)\max_i\{\|a_i\|_\infty + b_i\|x\|_\infty\}.\end{aligned} \tag{3.9}$$

This implies that $\|\mathfrak{T}x\|_\infty < \infty$, and so, $\mathfrak{T}$ maps $\mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r,+\infty)$ into $\mathrm{BPC}_{\mathrm{T}}[-r,+\infty)$.

By Torricelli–Barrow's classical result, the assumption (h_f) and the properties of $C(\cdot)x$ and $S(\cdot)x$, $(\mathfrak{T}x)(t)$ are differentiable, and for almost every $t \in (0, t_1] \cup \bigcup_{i=1}^N (s_i, t_{i+1}]$, it follows that:

$$(\mathfrak{T}x)'(t) := \begin{cases} AS(t)\phi(0) + C(t)\eta + \int_0^t C(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s, \\ \qquad\qquad \text{a.e. } t \in (0, t_1] \\ AS(t-s_i)\gamma_i(s_i, x(s_i)) + \int_{s_i}^t C(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s, \\ \qquad\qquad \text{a.e. } t \in \bigcup_{i=1}^N (s_i, t_{i+1}]. \end{cases} \tag{3.10}$$

By the same argument, we obtain that $\mathfrak{T}x(t)$ is twice differentiable and

$$(\mathfrak{T}x)''(t) := \begin{cases} \mathrm{AC}(t)\phi(0) + AS(t)\eta + \int_0^t AS(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s \\ \quad +f(t, x(t), x(\sigma(t))), \quad \text{a.e. } t \in (0, t_1] \\ \mathrm{AC}(t-s_i)\gamma_i(s_i, x(s_i)) + \int_{s_i}^t AS(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s \\ \quad +f(t, x(t), x(\sigma(t))), \quad \text{a.e. } t \in \bigcup_{i=1}^N (s_i, t_{i+1}]. \end{cases}$$

Consequently, it is not hard to find that

$$(\mathfrak{T}x)''(t) = A\mathfrak{T}x(t) + f(t, x(t), x(\sigma(t))), \qquad \text{a.e. } t \in \bigcup_{i=1}^N (s_i, t_{i+1}].$$

Since $\mathfrak{T}x|_{[-r,0]} = \phi$, $(\mathfrak{T}x)'(0) = \eta$ and $\mathfrak{T}x(t)|_{(t_i,s_i]} = \gamma_i(t, x(t))$, for all $i = 1, \ldots, N$, we obtain that the fixed points of $\mathfrak{T}$ are solutions of the differential system (P). □

Proposition 3.2 *The operator $\Gamma : BPC_{\mathrm{T}\cup\Sigma}[-r, +\infty) \to BPC_{\mathrm{T}\cup\Sigma}[-r, +\infty)$ is a ρ-contraction.*

Proof Let us note that, for any $x, y \in \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty)$:

$$|\Gamma x(t) - \Gamma y(t)| = \begin{cases} |\gamma_i(t, x(t)) - \gamma_i(t, y(t))|, & (t_i, s_i],\ i = 1, \ldots, N, \\ 0, & \text{otherwise.} \end{cases}$$

So, for all $t \in [-r, +\infty) \setminus \{\mathrm{T} \cup \Sigma\}$,

$$|\Gamma x(t) - \Gamma y(t)| \le \rho(|x(t) - y(t)|) \le \rho(\|x - y\|_\infty).$$

The claim follows by passing to the supremum in the left side of the inequality. □

Proposition 3.3 *The operator $T : BPC_{\mathrm{T}\cup\Sigma}[-r, +\infty) \to BPC_{\mathrm{T}\cup\Sigma}[-r, +\infty)$ defined by*

$$Tx(t) := \sum_{i=0}^N T_i x(t) \tag{3.11}$$

is completely continuous.

Proof In order to prove the continuity of T, let $t \in [-r, +\infty)$ be fixed and let $\{x_n\}$ be a sequence in $\mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty)$ converging to x. From (3.2) and (h_f), it follows that

$$\begin{aligned} |T_0x_n(t) - T_0x(t)| &\le |\int_0^t S(t-s)\left(f(s, x_n(s), x_n(\sigma(s))) - f(s, x(s), x(\sigma(s)))\right)\mathrm{d}s| \\ &\le \|f(\cdot, x_n(\cdot), x_n(\sigma(\cdot))) - f(\cdot, x(\cdot), x(\sigma(\cdot)))\|_1. \end{aligned}$$

In a similar way, by (3.3),

$$
\begin{aligned}
|T_i x_n(t) - T_i x(t)| &\le |C(t-s_i)\,(\gamma_i(s_i, x_n(s_i)) - \gamma_i(s_i, x(s_i)))\,| \\
&\quad + \| f(\cdot, x_n(\cdot), x_n(\sigma(\cdot))) - f(\cdot, x_n(\cdot), x_n(\sigma(\cdot)))\|_1 \\
&\le \rho(\|x_n - x\|_\infty) + \| f(\cdot, x_n(\cdot), x_n(\sigma(\cdot))) - f(\cdot, x(\cdot), x(\sigma(\cdot)))\|_1
\end{aligned}
$$

for any fixed $i = 1, \dots, N$.

From the above inequalities and (3.11), it is readily derived that

$$
\begin{aligned}
&\|T x_n(t) - T x(t)\| \le N\rho(\|x_n - x\|_\infty) + (N+1)\| f(\cdot, x_n(\cdot), x_n(\sigma(\cdot))) \\
&- f(\cdot, x(\cdot), x(\sigma(\cdot)))\|_1
\end{aligned} \tag{3.12}
$$

By using (h_f) again, we note that $f(t, x_n(t), x_n(\sigma(t)))$ converges to $f(t, x(t), x(\sigma(t)))$, and for a.e., $t \in [0, +\infty)$

$$
|f(t, x_n(t), x_n(\sigma(t)))| \le p(t) \sup_{n\in\mathbb{N}} \Psi(2\|x_n\|_\infty).
$$

Then, since $\{x_n\}$ is bounded, from the Lebesgue dominated convergence theorem, it follows that

$$
\| f(\cdot, x_n(\cdot), x_n(\sigma(\cdot))) - f(\cdot, x_n(\cdot), x_n(\sigma(\cdot)))\|_1 \to 0.
$$

This last relation, together with (3.12), implies that $\|T x_n - T x\|_\infty \to 0$ as $n \to \infty$. For a fixed $R > 0$, let

$$
B_R := \{u \in \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty) : \|u\|_\infty \le R\}.
$$

Corollary 2.8 ensures that, in order to prove the compactness of T, it is enough to show that $T(B_R)$ is a bounded set, quasi-equicontinuous and that for every $\epsilon > 0$, there exists $L > t_N$ such that

$$
\chi\left(T(B_R)_{|[L,+\infty)}\right) < \epsilon.
$$

By using (3.9), for all $x \in B_R$:

$$
|T x(t)| \le \|\phi\|_\infty + |\eta| + \Psi(2R)\|p\|_1 + \max_{k=1,\dots,N}(\|a_k\|_\infty + b_k R);
$$

which implies that $T(B_R)$ is bounded.

Let $\varepsilon > 0$. We first prove the quasi-equicontinuity on $[-r, +\infty) \setminus (\mathrm{T} \cup \Sigma)$ by dividing the proof into steps.

Step 1. If $\tau_1, \tau_2 \in [-r, 0]$ then

$$
|(T x)(\tau_1) - (T x)(\tau_2)| = |\phi(\tau_1) - \phi(\tau_2)|.
$$

Let τ_1, τ_2 be in $(0, t_1]$ with $\tau_2 \ge \tau_1$. Then,

$$
\begin{aligned}
|(T x)(\tau_1) - (T x)(\tau_2)| &= |(T_0 x)(\tau_1) - (T_0 x)(\tau_2)| \\
&\le |||C(\tau_1) - C(\tau_2)|||\,|\phi(0)| + |||S(\tau_1) - S(\tau_2)|||\,|\eta| \\
&\quad + \left| \int_0^{\tau_1} [S(\tau_1 - s) - S(\tau_2 - s)] f(s, x(s), x(\sigma(s)))\mathrm{d}s \right|
\end{aligned} \tag{3.13}
$$

$$
\quad + \left| \int_{\tau_1}^{\tau_2} S(\tau_2 - s) f(s, x(s), x(\sigma(s)))\mathrm{d}s \right|. \tag{3.14}
$$

Now, by property $(u5)$, we obtain that

$$S(\tau_2 - s) - S(\tau_1 - s) = [S(\tau_2) - S(\tau_1)]C(s) - [C(\tau_2) - C(\tau_1)]S(s).$$

It turns out that

$$\begin{aligned}
&\left|\int_0^{\tau_1} [S(\tau_1 - s) - S(\tau_2 - s)] f(s, x(s), x(\sigma(s)))\mathrm{d}s\right| \\
&\quad \le |||S(\tau_2) - S(\tau_1)||| \int_0^{\tau_1} |||C(s)||| |f(s, x(s), x(\sigma(s)))|\mathrm{d}s \\
&\qquad + |||C(\tau_2) - C(\tau_1)||| \int_0^{\tau_1} |||S(s)||| |f(s, x(s), x(\sigma(s)))|\mathrm{d}s \\
&\quad \le (|||S(\tau_2) - S(\tau_1)||| + |||C(\tau_2) - C(\tau_1)|||)\Psi(2R)\|p\|_1.
\end{aligned} \tag{3.15}$$

Substituting this into (3.13), we get

$$\begin{aligned}
|(Tx)(\tau_1) - (Tx)(\tau_2)| &= |(T_0x)(\tau_1) - (T_0x)(\tau_2)| \\
&= |||C(\tau_1) - C(\tau_2)||| |\phi(0)| + |||S(\tau_1) - S(\tau_2)||| |\eta| \\
&\quad + (|||S(\tau_2) - S(\tau_1)||| + |||C(\tau_2) - C(\tau_1)|||)\Psi(2R)\|p\|_1 \\
&\quad + \left|\int_{\tau_1}^{\tau_2} S(\tau_2 - s) f(s, x(s), x(\sigma(s)))\mathrm{d}s\right|.
\end{aligned}$$

Using (h_f), we also obtain that

$$\left|\int_{\tau_1}^{\tau_2} S(\tau_2 - s) f(s, x(s), x(\sigma(s)))\mathrm{d}s\right| \le \Psi(2R)\Big[\int_{\tau_1}^{\tau_2} p(s)\mathrm{d}s\Big]. \tag{3.16}$$

A further substitution into (3.14) gives that

$$\begin{aligned}
|(Tx)(\tau_1) - (Tx)(\tau_2)| &= |(T_0x)(\tau_1) - (T_0x)(\tau_2)| \\
&\le |||C(\tau_1) - C(\tau_2)|||(|\phi(0)| + \Psi(2R)\|p\|_1) \\
&\quad + |||S(\tau_1) - S(\tau_2)|||(|\eta| + \Psi(2R)\|p\|_1) \\
&\quad + \Psi(2R)\Big[\int_0^{\tau_2} p(s)\mathrm{d}s - \int_0^{\tau_1} p(s)\mathrm{d}s\Big].
\end{aligned}$$

So, calling $\vartheta = \Psi(2R)\|p\|_1 + \max\{|\phi(0)|, |\eta|\}$, we have if $\tau_1, \tau_2 \in [-r, t_1]$:

$$\begin{aligned}
|(Tx)(\tau_1) - (Tx)(\tau_2)| &= |(T_0x)(\tau_1) - (T_0x)(\tau_2)| \\
&\le \vartheta |||C(\tau_1) - C(\tau_2)||| + \vartheta |||S(\tau_1) - S(\tau_2)||| \\
&\quad + \Psi(2R)\Big[\int_0^{\tau_2} p(s)\mathrm{d}s - \int_0^{\tau_1} p(s)\mathrm{d}s\Big] \\
&\quad + |\phi(\tau_1) - \phi(\tau_2)|.
\end{aligned}$$

Step 2. Suppose that, for $i = 1, \ldots, N$, $\tau_1, \tau_2 \in (s_{i-1}, t_i]$ or $\tau_1, \tau_2 \in (s_i, +\infty)$ with $\tau_2 \ge \tau_1$. We have

$$\begin{aligned}
|(Tx)(\tau_1) - (Tx)(\tau_2)| &= |(T_ix)(\tau_1) - (T_ix)(\tau_2)| \\
&\le |||C(\tau_1 - s_i) - C(\tau_2 - s_i)||| |\gamma_i(s_i, x(s_i))|
\end{aligned} \tag{3.17}$$

$$+\left|\int_{s_i}^{\tau_1}[S(\tau_1-s)-S(\tau_2-s)]f(s,x(s),x(\sigma(s)))\mathrm{d}s\right| \tag{3.18}$$

$$+\left|\int_{\tau_1}^{\tau_2}S(\tau_2-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s\right|. \tag{3.19}$$

In (3.17), we use (u6) to derive

$$\begin{aligned}&|||C(\tau_1-s_i)-C(\tau_2-s_i)|||\,|\gamma_i(s_i,x(s_i))|\\&\quad\le(|||C(\tau_1)-C(\tau_2)|||+|||A|||\,|||S(\tau_1)-S(\tau_2)|||)(|a_i(s_i)|+b_i|x(s_i)|).\end{aligned}$$

In (3.18), we use (3.15) to get

$$\begin{aligned}&\left|\int_{s_i}^{\tau_1}[S(\tau_1-s)-S(\tau_2-s)]f(s,x(s),x(\sigma(s)))\mathrm{d}s\right|\\&\quad\le(|||S(\tau_2)-S(\tau_1)|||+|||C(\tau_2)-C(\tau_1)|||)\Psi(2R)\|p\|_1.\end{aligned}$$

Now using (3.16), we deduce

$$\begin{aligned}&|(Tx)(\tau_1)-(Tx)(\tau_2)|=|(T_ix)(\tau_1)-(T_ix)(\tau_2)|\\&\quad\le(|||C(\tau_1)-C(\tau_2)|||+|||A|||\,|||S(\tau_2)-S(\tau_1)|||)(|a_i(s_i)|+b_i|x(s_i)|)\\&\quad+(|||S(\tau_2)-S(\tau_1)|||+|||C(\tau_2)-C(\tau_1)|||)\Psi(2R)\|p\|_1\\&\quad+\Psi(2R)\left[\int_0^{\tau_2}p(s)\mathrm{d}s-\int_0^{\tau_1}p(s)\mathrm{d}s\right].\end{aligned}$$

Thus, in view of Step 1, we can redefine

$$\vartheta:=\Psi(2R)\|p\|_1+\max_{i=1,\dots,N}\{|\phi(0)|,|\eta|,\max\{1,|||A|||\}(|a_i(s_i)|+b_i|x(s_i)|)\}$$

to get that, for $t\in[-r,t_1]\cup\left(\bigcup_{i=1}^N(s_i,t_{i+1}]\right)$:

$$|(Tx)(\tau_1)-(Tx)(\tau_2)|\le\sum_{k=1}^4|\psi_k(\tau_1)-\psi_k(\tau_2)|,$$

where $\psi_1(t)=\vartheta C(t)$, $\psi_2(t)=\vartheta S(t)$, $\psi_3(t)=\Psi(2R)\int_0^t p(s)\mathrm{d}s$ and $\psi_4(t)=\phi(t)$.

By the uniform continuity of $C(\cdot)$, $S(\cdot)$ and ϕ, we have that there exists $\delta>0$ such that if $|\tau_1-\tau_2|<\delta$ for $\tau_1,\tau_2\in[-r,t_1]$ or $\tau_1,\tau_2\in(s_i,t_{i+1}]$, for some $i=1,\dots,N$ ($t_{N+1}=+\infty$), then

$$|(Tx)(\tau_1)-(Tx)(\tau_2)|<\epsilon.$$

Let us note that if $\tau_1<0<\tau_2$ with $|\tau_1-\tau_2|<\delta$, then

$$\begin{aligned}|(Tx)(\tau_1)-(Tx)(\tau_2)|&=|\phi(\tau_1)-(T_0x)(\tau_2)|\\&\le|\phi(\tau_1)-C(\tau_2)\phi(0)|+|S(\tau_2)-S(0)||\eta|\\&\quad+\left|\int_0^{\tau_2}S(\tau_2-s)f(s,x(s),x(\sigma(s)))\mathrm{d}s\right|\\&\le|\phi(\tau_1)-\phi(0)|+|||C(\tau_2)-C(0)|||\,|\phi(0)|\\&\quad+|||S(\tau_2)-S(0)|||\,|\eta|+\left|\Psi(2R)\int_0^{\tau_2}p(s)\mathrm{d}s\right|<\epsilon\end{aligned}$$

since $|\tau_2 - 0| < |\tau_2 - \tau_1| < \delta$. We also note that if $\tau_1, \tau_2 \in (t_i, s_i)$ for some $i = 1, \dots, N$, $(Tx)(\tau_1) = (Tx)(\tau_2) = 0$. So, T is quasi-equicontinuous on $[-r, +\infty) \setminus (\mathrm{T} \cup \Sigma)$.

To conclude the proof, note that since by (h_f), p is integrable. Thus, for an arbitrary $\varepsilon > 0$, there exists $L > t_N$ such that:

$$\int_L^{+\infty} p(s)\mathrm{d}s < \varepsilon.$$

So, for $x \in B_R$,

$$(Tx)_{|[L,+\infty)}(t) = C(t - s_N)\gamma_N(s_N, x(s_N)) + \int_{s_N}^t S(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s. \tag{3.20}$$

On the other hand,

$$K := \left\{ C(\cdot - s_N)\gamma_N(s_N, x(s_N)) + \int_{s_N}^L S(\cdot - s)f(s, x(s), x(\sigma(s)))\mathrm{d}s : x \in B_R \right\}$$

is a finite dimensional and bounded subset of $\mathrm{BPC}_{\mathrm{T}\cup\Sigma}[L, +\infty)$ and hence relatively compact.

Fix $t \geq L$; then,

$$\begin{aligned} &\left|(Tx)_{|[L,+\infty)}(t) - C(t - s_N)\gamma_N(s_N, x(s_N)) - \int_{s_N}^L S(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s\right| \\ &= \left|\int_L^t S(t-s)f(s, x(s), x(\sigma(s)))\mathrm{d}s\right| \leq \int_L^{+\infty} p(s)\mathrm{d}s < \varepsilon. \end{aligned}$$

It turns out that

$$T(B_R)_{|[L,+\infty)} \subset K + B_\varepsilon$$

and

$$\chi\left(T(B_R)_{|[L,+\infty)}\right) < \chi(K) + \chi(B_\varepsilon) \leq \varepsilon.$$

As a consequence, (2.4) holds and $T(B_R)$ is relatively compact. □

4 Main result

Theorem 4.1 *Assume the hypotheses* (h_A), (h_f), (h_σ), $(h_{\phi,\gamma})$. *Then, the problem* (P) *has at least one strong solution.*

Proof Our problem (P) can equivalently be reduced, by Proposition 3.1, to a fixed-point problem for the operator $\mathfrak{T}$. We shall make use of Theorem 2.18 and Lemma 2.19. We have already proved that:

- $\mathfrak{T}$ maps $\mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty)$ into $\mathrm{BPC}_{\mathrm{T}}[-r, +\infty)$ by Proposition 3.1.
- $\Gamma : \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty) \to \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty)$ is a ρ-contraction by Proposition 3.2.
- $T : \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty) \to \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty)$ is a completely continuous operator by Proposition 3.3.

It remains to prove that the set

$$\zeta(T + \Gamma) := \{x \in \mathrm{BPC}_{\mathrm{T}\cup\Sigma}[-r, +\infty) : x = \lambda\Gamma\left(\frac{x}{\lambda}\right) + \lambda T x for 0 < \lambda < 1\}$$

is bounded.

The idea is to divide the proof into four steps and proceed as in [6].

Assume $x = \lambda \Gamma(\frac{x}{\lambda}) + \lambda T x$ with $\lambda \in (0, 1)$. Then:

$$x(t) = \begin{cases} \lambda \phi(t), & t \in [-r, 0], \\ \lambda \sum_{i=1}^{N} \gamma_k \left(t, \frac{x(t)}{\lambda}\right) \chi_{(t_i, s_i]}(t), & t \in \bigcup_{i=1}^{N} (t_i, s_i], \\ \lambda \sum_{i=0}^{N} (T_i x)(t), & t \in (0, t_1] \cup \bigcup_{i=1}^{N} (s_i, t_{i+1}]. \end{cases} \tag{4.1}$$

Step 1. For $t \in [-r, 0)$, we have

$$|x(t)| = \lambda |\phi(t)| \leq \|\phi\|_\infty.$$

Step 2. For all $\xi \in [0, t_1]$, we have

$$|x(\xi)| = \lambda |(T_0 x)(\xi)| \leq |\phi(0)| + |\eta| + \int_0^\xi p(s) \Psi(|x(s)| + |x(\sigma(s))|) \mathrm{d}s. \tag{4.2}$$

So, for $\xi \in [-r, t_1]$,

$$\begin{aligned} |x(\xi)| &\leq \|\phi\|_\infty + |\phi(0)| + |\eta| \\ &\quad + \int_0^\xi p(s) \Psi(|x(s)| + |x(\sigma(s))|) \mathrm{d}s \\ &\leq 2\|\phi\|_\infty + |\eta| \\ &\quad + \int_0^\xi p(s) \Psi(|x(s)| + |x(\sigma(s))|) \mathrm{d}s \end{aligned} \tag{4.3}$$

Fix $t \in [-r, t_1]$ and let us define the function $\mu_x : [0, t_1] \to [0, +\infty)$ by

$$\mu_x(t) := \sup\{|x(\xi)| : -r \leq \xi \leq t\}.$$

As $\sigma(t) \leq t$ for $t \geq 0$, one has

$$\sup_{0 \leq s \leq t} |x(\sigma(s))| = \sup_{\sigma(0) \leq s \leq \sigma(t)} |x(s)| \leq \sup_{-r \leq s \leq t} |x(s)| = \mu_x(t).$$

Taking the supremum over $[-r, t]$ in the inequality (4.3), we obtain

$$\mu_x(t) \leq 2\|\phi\|_\infty + |\eta| + \int_0^t p(s) \Psi(2\mu_x(s)) \mathrm{d}s.$$

Denoting by $v_x(t)$ the right-hand side of the last inequality, we have that the function v_x is absolutely continuous,

$$c := v_x(0) = 2\|\phi\|_\infty + |\eta|$$

and $\mu_x(t) \leq v_x(t)$ for $t \in [0, t_1]$. Moreover, since Ψ is nondecreasing,

$$v_x'(t) = p(t) \Psi(2\mu_x(t)) \leq p(t) \Psi(2v_x(t)) \;\text{ a.e.}$$

This implies that for a.e. $t \in [0, t_1]$

$$\frac{v_x'(t)}{\Psi(2v_x(t))} \leq p(t),$$

and for any $t \in [0, t_1]$,

$$\int_0^t \frac{v_x'(s)}{\Psi(2v_x(s))} \mathrm{d}s \leq \int_0^t p(s) \mathrm{d}s =: G_t < \infty.$$

Note that v_x is absolutely continuous and nondecreasing. Since v_x' is nonnegative, it follows that

$$\int_{2c}^{2v_x(t)} \frac{\mathrm{d}s}{2\Psi(s)} \leq G_t. \tag{4.4}$$

We will show by contradiction that

$$\sup_{x\in\zeta(T+\Gamma)} \|v_x\|_\infty < \infty.$$

Indeed, suppose that there exists an unbounded sequence $\{v_n := v_{x_n}(t_n)\}$ and

$$\int_{2c}^{+\infty} \frac{\mathrm{d}s}{2\Psi(s)} = \lim_{n\to+\infty}\int_{2c}^{2v_n} \frac{\mathrm{d}s}{2\Psi(s)} < \infty \tag{4.5}$$

holds as a consequence.

To conclude, we note that inequality (4.5), together with condition (3.1), permits us to conclude that $(v_x)_{x\in\zeta(T+\Gamma)}$ is bounded by a constant Δ_1 depending only on the functions Ψ, p, η and ϕ.

Step 3. For $t \in (t_1, s_1]$, we have

$$\begin{aligned}|x(t)| &= \lambda\left|\gamma_1\left(t, \frac{x(t)}{\lambda}\right)\right| \\ &\leq \lambda|a_1(t)| + b_1|x(t)| \leq |a_1(t)| + b_1|x(t)|\end{aligned}$$

so,

$$|x(t)| \leq \frac{1}{1-b_1}\|a_1\|_\infty := \delta_1.$$

This implies that for $t \in [-r, s_1]$,

$$|x(t)| \leq \tilde{\Delta}_1 := \Delta_1 + \delta_1.$$

Step 4. Let $\xi \in (s_1, t_2]$, we have

$$|x(\xi)| = \lambda|(T_1x)(\xi)| \leq \|a_1\|_\infty + b_1|x(s_1)| + \int_{s_1}^{\xi} p(s)\Psi(|x(s)| + |x(\sigma(s))|)\mathrm{d}s$$

and so, for $\xi \in [-r, t_2]$,

$$|x(\xi)| \leq \tilde{\Delta}_1 + \|a_1\|_\infty + b_1|x(s_1)| + \int_{s_1}^{\xi} p(s)\Psi(|x(s)| + |x(\sigma(s))|)\mathrm{d}s. \tag{4.6}$$

We reason as for (4.3) in Step 1. Let us fix $t \in [-r, t_2]$ and define the function $\mu_x : [0, t_2] \to [0, +\infty)$ by

$$\mu_x(t) := \sup\{|x(\xi)| : -r \leq \xi \leq t\}.$$

Observe now that μ_x is not necessarily continuous at t_1 and s_1, but the right limits exist. For $\sigma(t) \leq t$ for $t \geq 0$, one has

$$\sup_{0\leq s\leq t}|x(\sigma(s))| = \sup_{\sigma(0)\leq s\leq\sigma(t)}|x(s)| \leq \sup_{-r\leq s\leq t}|x(s)| = \mu_x(t).$$

So, by taking the supremum over $[-r, t]$ in the inequality (4.6), we obtain

$$\mu_x(t) \le \tilde{\Delta}_1 + \|a_1\|_\infty + b_1\mu_x(t) + \int_0^t p(s)\Psi(2\mu_x(s))\mathrm{d}s \;\Rightarrow$$
$$\mu_x(t) \le \frac{1}{1-b_1}\left(\tilde{\Delta}_1 + \|a_1\|_\infty + \int_0^t p(s)\Psi(2\mu_x(s))\mathrm{d}s\right).$$

Denoting by $v_x(t)$ the right-hand side of the last inequality, we have that the function v_x is continuous,

$$c := v_x(0) = \frac{1}{1-b_1}\left(\tilde{\Delta}_1 + \|a_1\|_\infty\right)$$

and $\mu_x(t) \le v_x(t)$ for $t \ge 0$. Moreover, since Ψ is nondecreasing, for $t \ne t_1, t \ne s_1$,

$$v_x'(t) = p(t)\Psi(2\mu_x(t)) \le p(t)\Psi(2v_x(t)).$$

This implies that

$$\frac{v_x'(t)}{\Psi(2v_x(t))} \le p(t), \qquad t \ne t_1,\, t \ne s_1$$

and so, for any $b > 0$,

$$\int_0^b \frac{v_x'(t)}{\Psi(2v_x(t))}\mathrm{d}t \le \int_0^b p(t)\mathrm{d}t := \Gamma_b < \infty.$$

Since v_x' is a continuous function for all $t \ne t_1,\ t \ne s_1$, we have

$$\int_{2c}^{2v_x(b)} \frac{\mathrm{d}s}{2\Psi(s)} \le \Gamma_b.$$

This, together with condition (3.1), permits us to conclude that v_x is bounded by a constant Δ_2 depending only on the functions $a_1, a_2,\ \Psi,\ p,\ \eta,\ \phi$ and the values b_1, b_2.

By repeating the last two steps N-times, we obtain that if $x \in \zeta(T+\Gamma)$ and $t \in [-r, +\infty)$, then there exits $\Delta_{N+1} := \Delta_{N+1}(\eta, \phi, \Psi, p, (a_i)_{i=1,\dots,N}, (b_i)_{i=1,\dots,N})$ such that

$$|x(t)| \le \mu_x(t) \le v_x(t) \le \Delta_{N+1} < +\infty.$$

This completes the proof. □

As a side result of Theorem 4.1, we have proved that the obtained solution is bounded in $[-r, +\infty)$. We stress that hypothesis (h_f) can be lowered at the expense of losing the boundedness of the solutions, as the following corollary shows.

Corollary 4.2 *Assume the hypotheses (h_A), (h_σ), $(h_{\phi,\gamma})$. Assume also the following condition is satisfied:*

(h_f^*) $f : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is such that a nonnegative function $p^* \in L^1_{\mathrm{loc}}[0, +\infty)$ and a continuous nondecreasing function $\Psi : [0, +\infty) \to [1, +\infty)$ exist for which

$$|f(t, x, y)| \le p^*(t)\Psi(|x| + |y|),\ t \ge 0,\ x, y \in \mathbb{R}^n,$$

and

$$\int_0^\infty \frac{ds}{\Psi(s)} = +\infty. \tag{4.7}$$

Then, the problem (P) admits at least one (not necessarily bounded) strong solution.

Proof The proof follows a standard argument which we sketch in the sequel.

Fix $n \in \mathbb{N}$ such that $n \geq t_N$ and let

$$i_n(t) := \begin{cases} 1, & \text{if } t \in [0, n] \\ 0, & \text{otherwise} \end{cases}$$

be the indicator function of the set $[0, n]$.

Let $f_n(t, x, y) := i_n(t) f(t, x, y)$ and $p_n(t) := i_n(t) p^*(t)$. Then, it holds that f_n satisfies the condition (h_f), with $p = p_n \in L^1[0, +\infty)$.

From Theorem 4.1, it follows that the problem

$$(P_n) \quad \begin{cases} x''(t) = Ax(t) + f_n(t, x(t), x(\sigma(t))), & t \in (0, t_1] \cup \bigcup_{i=1}^N (s_i, t_{i+1}] \\ x(t) = \gamma_i(t, x(t)), & t \in \bigcup_{i=1}^N (t_i, s_i], \\ x(t) = \phi(t), \quad t \in [-r, 0], & x'(0) = \phi'(0) := \eta \end{cases}$$

has a solution x_n. For $t \in [0, +\infty)$, let $n(t) := \min_{n \in \mathbb{N}}\{n \geq t\}$ and define $x : [-r, +\infty) \to \mathbb{R}^n$ by

$$x(t) := x_{n(t)}(t).$$

Then, it easily follows that x is a solution of the problem (P). $\square$

5 Conclusions

Theorem 4.1 differs from previous results on the subject (see [12] and [13]) in several ways. At first, we analyzed a second-order system with non-instantaneous impulses, while the previous literature concerns only the first-order case. On the other hand, the problem is stated and studied on unbounded intervals and with an ad hoc technique. Lastly, the delay introduced exploits the role played by the non-instantaneous impulses.

In our opinion, the results can be of interest for modeling the actions of chemotherapeutic drugs. Indeed, in this case:

1. Second-order differential systems naturally arise in models where the spatial components play an important role (e.g., cancers), as well as in the so-called two-compartment models.
2. By using an unbounded interval, we can shape a multiscale model, where the jump at each point of discontinuity may represent the diffusion of the drug in the bloodstream (which occurs in few minutes), the non-instantaneous part represents the absorption of the drugs by the cells (which can be measured in days), while the differential system defined on the unbounded interval models the whole trend from the first chemotherapeutic session up to the end of the cure, which may last years.
3. By following [19], the delay can shape the behavior of the affected cells. Indeed, we stress that in case of chemotherapy, the cells react to the drugs by committing "suicide" only after the DNA checking cycle, i.e., with a delay.

References

1. Banas, J., Goebel, K.: Measure of noncompactness in Banach spaces. In: Lecture Notes in Pure and Applied Mathematics, vol. 60. M. Dekker, New York-Basel (1980)
2. Bartle, R.G.: On compactness in functional analysis. Trans. Am. Math. Soc. **79**(1), 35–57 (1955)

3. Cai, G., Bu, S.: Krasnoselskii-type fixed point theorems with applications to Hammerstein integral equations in L^1 spaces. Math. Nachr. **286**(14–15), 1452–1465 (2013)
4. Campbell, S.A., Edwards, R., van den Driessche, P.: Delayed coupling between two neural network loops. SIAM J. Appl. Math. **65**, 316–335 (2004)
5. Cooke, K.L., van den Driessche, P., Zou, X.: Interaction of maturation delay and nonlinear birth in population and epidemic models. J. Math. Biol. **39**, 332–352 (1999)
6. Dauer, J.P., Balachandran, K.: Existence of solutions of nonlinear neutral integro-differential equations in Banach spaces. J. Math. Anal. Appl. **251**, 93–105 (2000)
7. Erneux, T.: Applied Delay Differential Equations. Springer, Berlin (2009)
8. Fattorini, H.O.: Second Order Linear Differential Equations in Banach Spaces, vol. 108. North-Holland Mathematics Studies, Amsterdam (1985)
9. Garcia-Falset, J.: Existence of fixed points for the sum of two operators. Math. Nachr. **283**(12), 1736–1757 (2010)
10. Goldstein, J.A.: Semigroups of Linear Operators and Applications. Oxford University Press, New York (1985)
11. Graef, J.R., Ouahab, A.: Global existence and uniqueness results for impulsive functional differential equations with variable times and multiple delays. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **16**(1), 27–40 (2009)
12. Hernandez, E., Rabello, M., Henriquez, H.R.: Existence of solutions for impulsive partial neutral functional differential equations. J. Math. Anal. Appl. **331**, 1135–1158 (2007)
13. Hernandez, E., O'Regan, D.: On a new class of abstract impulsive differential equations. Proc. Am. Math. Soc. **141**(5), 1641–1649 (2013)
14. Hernandez, E., Tanaka Aki, S.M.: Global solutions for abstract impulsive differential equations. Nonlinear Anal. Theory Methods Appl. **72**(3), 1280–1290 (2010)
15. Krasnosel'skii, M.A.: Some problems of nonlinear analysis. Uspekhi Mat. Nauk **9**(3), 57–114 (1954)
16. Lakshmikantham, V., Bainov, D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
17. Liu, Y., Li, Z.: Krasnoselskii type fixed point theorems and applications. Proc. Am. Math. Soc. **136**(4), 1213–1220 (2008)
18. Muglia, L., Pietramala, P.: Second-order impulsive differential equations with functional initial conditions on unbounded intervals. J. Funct. Spaces **2013**, 479049-1–479049-9 (2013)
19. Perelson, A.S., Neumann, A.U., Markowitz, M., Leonard, J.M., Ho, D.D.: HIV $-$ 1 dynamics in vivo: virion clearance rate, infected cells life-span, and viral generation time. Science **271**, 1582–1586 (1996)
20. Perestyuk, N.A., Samoilenko, A.M.: Differential Equations with Impulse Effect. Visca Skola, Kiev (1987)
21. O'Regan, D.: Fixed-point theory for the sum of two operators. Appl. Math. Lett. **9**(1), 1–8 (1996)
22. Stamova, I.: Stability Analysis of Impulsive Functional Differential Equations. Walter de Gruyter, Berlin (2009)
23. Shen, J., Liu, X.: Global existence results for impulsive differential equations. J. Math. Anal. Appl. **314**(2), 546–557 (2006)
24. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung. **32**(1–2), 75–96 (1978)
25. Zhao, T.: Global periodic-solutions for a differential delay system modeling a microbial population in the chemostat. J. Math. Anal. Appl. **193**, 329–352 (1995)

A HIERARCHICAL APPROACH TO FIXED POINT PROBLEMS FOR UNIFORMLY ASYMPTOTICALLY REGULAR SEQUENCES.

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Abstract. We study the hierarchical iterative scheme defined by $x_1 \in H$ and $x_{n+1} = \alpha_n x_n + (1-\alpha_n)(I - \mu_n D)W_n x_n + \alpha_n \beta_n (S - I)x_n$ $(n \geq 1)$, involving a UAR sequence $(W_n)_{n\in\mathbb{N}}$. We prove that the limit point depends from the value $\tau := \lim_{n\to\infty} \alpha_n \beta_n / \mu_n$.

1. Introduction and Motivations

The approximation of solutions for Variational Inequality Problems (VIPs) had always been a wide and catalyzing research area. This is due, among other motivations, to the fact that many real-world problems can be modeled as VIPs. To recognize eligible problems, we refer to the Cabot's paper [8], Deutsch and Yamada's paper [15], Combettes and Hirstoaga's paper [13], Yamada's paper [33] on monotone inclusions, Solodov's paper [27]) on convex optimization and [11, 21, 22, 30, 32] on quadratic minimization over fixed point.

It is well known that if D is a monotone operator and C is a closed and convex subset of the Hilbert space H, then the existence of solutions for the (VIP)

$$\text{Find } x^* \in C \text{ such that } \langle Dx^*, y - x\rangle \geq 0, \qquad \forall y \in C \tag{1.1}$$

is not guaranteed. Nevertheless, under the assumption that A is continuous, the set of solution is nonempty.

In this paper we consider a strongly monotone and Lipschitzian operator D, i.e. D satisfies

$$\langle Dx - Dy, x - y\rangle \geq \sigma^2 \|x - y\|^2 \text{ and } \|Dx - Dy\| \leq L\|x - y\|,$$

for any $x, y \in C$. By this assumption, the set of solution for the problem (1.1) is reduced to a unique point (see Browder and Petryshyn [6] or Deimling [14]).

Despite the fact that the above mentioned hypothesis may appear tricky, several examples of (VIPs) involving strongly monotone and Lipschitz operators can be easily found in the literature.

For instance, if we fix $u \in H$ and set $Dx = x - u$, the inequality in (1.1) becomes

$$\langle x^* - u, y - x\rangle \geq 0, \qquad \forall y \in C.$$

In this case x^* is the solution of the minimum distance problem

$$\|x^* - u\| = \min_{x\in C} \|x - u\|.$$

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In 2006, Marino and Xu [22], studied the problem of minimizing a quadratic function over the set $Fix(T)$ of the fixed points of a nonexpansive mapping T on a real Hilbert space H, i.e.

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x)$$

where h is a potential function of a contraction f (i.e., $h'(x) = \gamma f(x)$ for $x \in H$) and A is a strongly positive linear and bounded operator.
We note that $x^* \in Fix(T)$ solves the minimization problem if and only if

$$\langle (A - \gamma f)x^*, x^* - z\rangle \geq 0, z \in Fix(T),$$

where $D = (A - \gamma f)$ is strongly monotone and Lipschitzian.

Reich and Xu in [26] considered the constrained least squares problem:

$$\min_{x\in\bigcap_{i\in I} C_i} \frac{1}{2}\|Ax - b\|^2, \tag{1.2}$$

where A is a bounded linear operator on H, $\{C_i\}_{i\in I}$ is a family of closed and convex subsets of H and $b \in H$ is fixed. Since this problem may be ill-posed (i.e. it may have more than one solution), its Tichonov regularization

$$\min_{x\in C} \frac{1}{2}(\|Ax - b\|^2 + \epsilon\|x\|^2) \tag{1.3}$$

is introduced and studied by translating (1.3) into te following:

Find $x^* \in \cap_{i\in I} C_i$ such that $\langle (A^*A + \varepsilon I)x_\varepsilon - A^*b, x - x_\varepsilon\rangle \geq 0, \quad x \in \cap_{i\in I} C_i,$

where A^* is the self adjoint operator of A and $(A^*A + \varepsilon I)$ becomes a strongly monotone and Lipschitzian operator.

We point out that (1.2) belongs from an interesting case of (VIPs), i.e. the class of convex feasibility problems (CFPs). The convex feasibility problem can be stated as follows,

$$\min_{x\in\cap_{i\in I} C_i} \Phi(x),$$

where Φ represents a cost-function to be minimized and $\{C_i\}_{i\in I}$ are convex and closed property sets, representing the constraints.
Firstly applied in Optimization, (CFPs) gained importance in different branches of sciences and they are currently used to model problems arising in image reconstruction, linear prediction theory and signal processing (see [5]), among others. For an exhaustive introduction on the subject, one can refer to [3].
Apart from the linear case, where $\{C_i\}_{i\in I}$ is a family of half-planes, it can be convenient to threat each C_i as the fixed point set $Fix(T_i)$ for a nonexpansive operator T_i, so that $\cap_{i_I} C_i$ represents the set of common fixed points of a family of nonexpansive mappings. The solution is then reached by means of iterations involving the family $\{T_i\}_{i\in I}$ (see [1, 2, 3, 4, 10, 18, 21] and references therein).
In [20] and [21], an unifying approach to the problem had been proposed, by introducing the $UAR-$class of procedures. This approach consists into introduce a sequence of mappings $(W_n)_{n\in\mathbb{N}}$ that preserves the fixed points of the family $(T_i)_{i\in I}$ and such that the following conditions are satisfied:

(h1) $W_n : H \to H$ are nonexpansive mappings, uniformly asymptotically regular on bounded subsets $B \subset H$, i.e.

$$\lim_{n\to\infty} \sup_{x\in B} \|W_{n+1}x - W_n x\| = 0,$$

(h2) it is possible to define a nonexpansive mapping $W : H \to H$, with $Wx := \lim_{n\to\infty} W_n x$ such that if $F := \bigcap_{n\in\mathbb{N}} Fix(W_n) \neq \emptyset$ then $Fix(W) = F$.

Classes of $UAR-$procedures can be widely found in the existing literature, as the following examples show.

Example 1.1 have been introduced by Atsushiba and Takahashi, while Shimoji and Takahashi developed the construction in example 1.2. Both results had been essential to detect the role of (h1) and (h2). We emphasize that the introduction of families of auxiliary mappings, essentially due to Professor Takahashi, represents the starting point of wide and deep line of research.

Example 1.1 (Atsushiba-Takahashi, [1] 1999)**.** *Let E be a strictly convex Banach space. Let $\Sigma = \{T_i\}_{i=1}^N$ be a family of nonexpansive mappings from C to C with $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Let $(\Lambda_n)_{n\in\mathbb{N}} = (\lambda_{1,n}, \dots, \lambda_{N,n}) \subset (0,1)^N$ such that $\lambda_{i,n} \to \lambda_i \in (0,1)$, as $n \to \infty$, for all $i = 1, \dots, N$.*
Let

$$\begin{cases} U_{1,n} = \lambda_{1,n}T_1 + (1-\lambda_{1,n})I \\ U_{2,n} = \lambda_{2,n}T_2U_{1,n} + (1-\lambda_{2,n})I \\ \vdots \\ W_n = W_{\Sigma,\Lambda_n} := U_{N,n} = \lambda_{N,n}T_NU_{N-1,n} + (1-\lambda_{N,n})I. \end{cases} \tag{1.4}$$

The mappings $(W_n)_{n\in\mathbb{N}}$ satisfies (h1) and (h2) and the proof of this fact is contained in Lemma 3.1 and Lemma 3.2 of [1] *(see* [12, 21] *for details). In particular $\bigcap_{i=1}^N Fix(T_i) = \bigcap_{n\in\mathbb{N}} Fix(W_n) = Fix(W)$.*

Example 1.2 (Shimoji-Takahashi, [28] 2001)**.** *Let X be a strictly convex Banach space and $C \subset X$ closed and convex.*
Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C to C with $\bigcap_{n\in\mathbb{N}} Fix(T_n) \neq \emptyset$. Let $\Lambda := (\lambda_n)_{n\in\mathbb{N}} \subset (0,b] \subset (0,1)$. Let consider the following construction:

$$\begin{aligned} &U_{n,n+1} := I, \\ &U_{n,n} := \lambda_nT_nU_{n,n+1} + (1-\lambda_n)I, \\ &\vdots \\ &U_{n,k} := \lambda_kT_kU_{n,k+1} + (1-\lambda_k)I, \\ &\vdots \\ &U_{n,2} := \lambda_2T_2U_{n,3} + (1-\lambda_2)I, \\ &W_n \equiv U_{n,1} := \lambda_1T_1U_{n,2} + (1-\lambda_1)I. \end{aligned} \tag{1.5}$$

The Shimoji-Takahashi's approach in (1.5) satisfies (h1) and (h2) and the proof easily follows by Lemma 3.1 and 3.2 in [28] *(see* [20]*). Also in this case $\bigcap_{i\in\mathbb{N}} Fix(T_i) = \bigcap_{n\in\mathbb{N}} Fix(W_n) = Fix(W)$.*

Example 1.3. *Let A be a δ-inverse strongly monotone operator, i.e. satisfies:*

$$\langle Ax - Ay, x - y\rangle \geq \delta\|Ax - Ay\|^2.$$

with $A^{-1}0 \neq \emptyset$ *(see* [29]*). Let* $(\lambda_n)_{n\in\mathbb{N}} \subset (0, 2\delta]$ *be a sequence converging to* $\lambda \in (0, 2\delta]$.

Then $W_n := (I - \lambda_n A)$ *is a family of nonexpansive mappings (see page 419 in* [29]*) with common fixed point set* $F = A^{-1}0$, *satisfying (h1) and (h2).*

Our last example is due to Gu et.al. in [17].

Example 1.4. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of nonexpansive mappings with at least one common fixed point. Let* $(\lambda_n)_{n\in\mathbb{N}} \subset (0, 1)$ *a strictly decreasing sequence such that* $\lambda_0 = 1$ *and* $\displaystyle\sum_{i=1}^{\infty}(\lambda_{i-1} - \lambda_i) < \infty$. *Let, for every* $n \geq 1$

$$W_n x = \frac{\sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)T_i x}{1 - \lambda_n}$$

Note that all W_n *are nonexpansive mappings such that,*

$$\begin{aligned}
\|W_{n+1}x - W_n x\| &\leq \left\| \frac{\sum_{i=1}^{n+1}(\lambda_{i-1} - \lambda_i)T_i x}{1 - \lambda_{n+1}} - \frac{\sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)T_i x}{1 - \lambda_n} \right\| \\
&\leq \left| \frac{1}{1 - \lambda_{n+1}} - \frac{1}{1 - \lambda_n} \right| \sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)\|T_i x\| + \frac{(\lambda_n - \lambda_{n+1})\|T_{n+1}x\|}{1 - \lambda_{n+1}} \\
&= \frac{\lambda_n - \lambda_{n+1}}{1 - \lambda_{n+1}} \left(\frac{\sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)\|T_i x\|}{1 - \lambda_n} + \|T_{n+1}x\| \right)
\end{aligned}$$

So, if x *lies in a bounded set, by an opportune* $M > 0$

$$\begin{aligned}
\|W_{n+1}x - W_n x\| &\leq \frac{\lambda_n - \lambda_{n+1}}{1 - \lambda_{n+1}} M \left(\frac{\sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)}{1 - \lambda_n} + 1 \right) \\
&= 2M \frac{\lambda_n - \lambda_{n+1}}{1 - \lambda_{n+1}}
\end{aligned}$$

Thus, since $\displaystyle\sum_{i=1}^{\infty}(\lambda_{i-1} - \lambda_i) < \infty$, $\displaystyle\sum_{n=1}^{\infty}\|W_{n+1}x - W_n x\|$ *converges and this guarantees that* $(W_n)_{n\in\mathbb{N}}$ *is pointwise convergent and uniformly asymptotically regular on the bounded subset of* C. *Notice that, since* $\dfrac{\lambda_{i-1} - \lambda_i}{1 - \lambda_n} \in [0, 1]$, *for all* i *and* $\dfrac{\sum_{i=1}^{n}(\lambda_{i-1} - \lambda_i)}{1 - \lambda_n} = 1$ *for all* n *then* $\bigcap_{i\in\mathbb{N}} Fix(T_i) = \bigcap_{n\in\mathbb{N}} Fix(W_n) = Fix(W)$ *by Lemma 3, page 257, proved in* [7].

Here we consider a sequence of mapping $(W_n)_{n\in\mathbb{N}}$ with common fixed points to emphasize the role played by (h1) and (h2) to the convergence of the method.

The hierarchical approach draws on from Moudafi in [25], where the following explicit algorithm is carried out:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma_n)T x_n). \tag{1.6}$$

In the same paper, it is proved a weak convergence result of the method (1.6) to a solution of

$$\langle (I - S)x^*, y - x^* \rangle \geq 0, \ \forall y \in Fix(T). \tag{1.7}$$

Following [23] and [24], Cianciaruso et al., in [9] studied the following scheme:

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \quad n \geq 1 \end{cases} \tag{1.8}$$

and proved three convergence results to solutions of variational inequality problems. Inspired by the above mentioned results, we investigate the convergence of the iterative method generated by $x_1 \in H$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)(I-\mu_n D)W_n x_n + \alpha_n\beta_n(S-I)x_n, \quad n \geq 1 \tag{1.9}$$

where D is a strongly monotone and Lipschitzian operator, W_n is a sequence of mappings satisfying $(h1)$ and $(h2)$ and S is a nonexpansive mapping.
In particular we will show that the iterative method (3.1) converges to a solution of a variational inequality problem that involves the operators D and $(I-S)$ and that such convergence depends by the value

$$\tau := \lim_{n\to\infty} \frac{\alpha_n\beta_n}{\mu_n}.$$

2. Preliminar results

We recall some general results about Hilbert spaces and monotone operators.

Lemma 2.1. *For all $x, y \in H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$
$$\|tx+(1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$$

If K is closed convex subset of a real Hilbert space H, the metric projection $P_K : H \to K$ is the mapping defined as follows: for each $x \in H$, $P_K x$ is the only point in K with the property

$$||x - P_K x|| = \inf_{y\in K} ||x-y||\,.$$

Lemma 2.2. [6] *Let K be a nonempty closed convex subset of a real Hilbert space H and let P_K be the metric projection from H onto K. Given $x \in H$ and $z \in K$, $z = P_K x$ if and only if*

$$\langle x-z, y-z\rangle \leq 0 \quad \forall\, y \in K\,.$$

Lemma 2.3. [16] *Let $W : C \to C$ be a nonexpansive mapping. Then, for all $x, y \in C$:*

- *the mapping $(I-W)$ is $\frac{1}{2}$-inverse strongly monotone that is*

$$\langle (I-W)x - (I-W)y, x-y, \rangle \geq \frac{1}{2}\|(I-W)x-(I-W)y\|^2.$$

- *The operator $(D+(I-W))$ is a σ-strongly monotone operator and $\left(\frac{\sigma}{L}+\frac{1}{2}\right)$-inverse strongly monotone operator.*

Lemma 2.4. [31] *Let $D : H \to H$ be a $\sigma-$strongly monotone and L-lipschitazian operator. If $\mu < \dfrac{2\sigma}{L^2}$, $\rho = \dfrac{2\sigma-\mu L^2}{2}$ and $(\mu_n)_{n\in\mathbb{N}} \subset (0,\mu]$ then:*

$$\|(I-\mu_n D)x - (I-\mu_n D)y\| \leq (1-\mu_n\rho)\|x-y\|$$

i.e. $(I-\mu_n D)$ is a $(1-\mu_n\rho)$-contraction.

Finally, we conclude this section with a lemma due to Xu on real sequences which has a fundamental role in the sequel.

Lemma 2.5. [30] *Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ and $(\delta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that,

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Properties and Convergence of the Iterative Algorithm

Let (W_n) be a sequence of mappings with common fixed point set $F := \cap_{n\in\mathbb{N}} Fix(W_n) \neq \emptyset$ and satisfying the properties

(h1) $W_n : H \to H$ are nonexpansive mappings, uniformly asymptotically regular on bounded subsets $B \subset H$, i.e.

$$\lim_{n\to\infty} \sup_{x\in B} \|W_{n+1}x - W_n x\| = 0,$$

(h2) it is possible to define a nonexpansive mapping $W : H \to H$, with $Wx := \lim\limits_{n\to\infty} W_n x$ such that if $F := \bigcap_{n\in\mathbb{N}} Fix(W_n) \neq \emptyset$ then $Fix(W) = F$.

Let $D : H \to H$ be a σ-strongly monotone and L-lipschitzian operator on H, i.e. D satisfies

$$\langle Dx - Dy, x - y\rangle \ge \sigma^2 \|x-y\|^2 \text{ and } \|Dx - Dy\| \le L\|x-y\|$$

and let $S : H \to H$ be a nonexpansive mapping.
Moreover, fix $\alpha \in (0,1)$ and $\mu \in (0, \frac{2\sigma}{L^2})$ and consider the sequences $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\alpha]$, $(\beta_n)_{n\in\mathbb{N}} \subset (0,1)$ and $(\mu_n)_{n\in\mathbb{N}} \subset (0,\mu)$, with the property that there exists the limit

$$\tau := \lim_{n\to\infty} \frac{\alpha_n \beta_n}{\mu_n}.$$

We introduce the iterative scheme defined by $x_1 \in H$

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) B_n W_n x_n + \alpha_n \beta_n (S - I) x_n, \qquad n \ge 1, \tag{3.1}$$

where $B_n := (I - \mu D)$ and we will prove that the convergence of the iterated sequence mainly depends on τ.

Lemma 3.1. *Assume that either*

$$\tau < +\infty, \tag{H1}$$

(i) $F \cap Fix(S) \neq \emptyset$, *or*
(ii) S *has bounded range.*

Then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Proof. Let $z \in F$ be a fixed element.
Assume that (H1) holds and, as a consequence, that $\frac{\alpha_n\beta_n}{\mu_n}$ is bounded from above by some constant $\gamma > 0$. We have then,

$$\begin{aligned}
\|x_{n+1}-z\| &\leq \|\alpha_n(x_n-z)+(1-\alpha_n)(B_nW_nx_n-B_nz)+(1-\alpha_n)(B_nz-z)\\
&\quad +\alpha_n\beta_n(Sx_n-x_n)\|\\
&= \|\alpha_n(x_n-z)+(1-\alpha_n)(B_nW_nx_n-B_nz)+(1-\alpha_n)(B_nz-z)\\
&\quad +\alpha_n\beta_n(Sx_n-Sz)+\alpha_n\beta_n(Sz-z)+\alpha_n\beta_n(z-x_n)\|\\
&= \|\alpha_n(1-\beta_n)(x_n-z)+(1-\alpha_n)(B_nW_nx_n-B_nz)+(1-\alpha_n)(B_nz-z)\\
&\quad +\alpha_n\beta_n(Sx_n-Sz)+\alpha_n\beta_n(Sz-z)\|\\
&\leq \alpha_n(1-\beta_n)\|x_n-z\|+(1-\alpha_n)\|B_nW_nx_n-B_nz\|+(1-\alpha_n)\|B_nz-z\|\\
&\quad +\alpha_n\beta_n\|Sx_n-Sz\|+\alpha_n\beta_n\|Sz-z\|\\
&\leq \alpha_n\|x_n-z\|+\alpha_n\beta_n\|Sz-z\|\\
&\quad +(1-\alpha_n)(1-\mu_n\rho)\|x_n-z\|+(1-\alpha_n)\mu_n\|Dz\|\\
&\leq (1-(1-\alpha_n)\mu_n\rho)\|x_n-z\|+\mu_n\|Dz\|+\alpha_n\beta_n\|Sz-z\| \qquad (3.2)\\
&\leq (1-(1-\alpha_n)\mu_n\rho)\|x_n-z\|+\mu_n(\|Dz\|+\gamma\|Sz-z\|)
\end{aligned}$$

So, by an inductive process, one can see that

$$\|x_n-z\| \leq \max\left\{\|x_1-z\|, \frac{\|Dz\|+\gamma\|Sz-z\|}{\rho}\right\}.$$

so the claim follows.

Suppose that (i) holds. In this case, z can be chosen in $Fix(S)\cap F$, so that $\|Sz-z\|=0$. By using (3.2), it is easily derived that

$$\|x_{n+1}-z\| \leq (1-(1-\alpha_n)\mu_n\rho)\|x_n-z\|+\mu_n\|Dz\|$$

and

$$\|x_n-z\| \leq \max\left\{\|x_1-z\|, \frac{\|Dz\|}{\rho}\right\}.$$

Lastly, note that

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\alpha_n\left((1-\beta_n)(x_n-z)+\beta_n(Sx_n-z)\right)\\
&\quad +(1-\alpha_n)(B_nW_nx_n-B_nz+B_nz-z)\|\\
&\leq \alpha_n((1-\beta_n)\|x_n-z\|+\beta_n\|Sx_n-z\|)\\
&\quad +(1-\alpha_n)\left((1-\mu_n\rho)\|x_n-z\|+\mu_n\rho\frac{\|Dz\|}{\rho}\right)\\
&\leq \alpha_n\max\{\|x_n-z\|,\|Sx_n-z\|\}+(1-\alpha_n)\max\left\{\|x_n-z\|,\frac{\|Dz\|}{\rho}\right\}\\
&\leq \max\left\{\|x_n-z\|,\|Sx_n-z\|,\frac{\|Dz\|}{\rho}\right\}.
\end{aligned}$$

If we assume that (ii) holds, i.e. that

$$\|Sx_n-z\| \leq M,$$

for some constant $M > 0$, then it is promptly derived that

$$\|x_n - z\| \leq \max\left\{\|x_1 - z\|, M, \frac{\|Dz\|}{\rho}\right\}.$$

□

Lemma 3.2. *Assume that $(x_n)_{n\in\mathbb{N}}$ is bounded and asymptotically regular; if $\mu_n \to 0$ and $\alpha_n\beta_n \to 0$ as $n \to +\infty$ then:*

1. $\lim_{n\to\infty} \|x_n - W_n x_n\| = 0$;
2. *the set of weak cluster points of $(x_n)_{n\in\mathbb{N}}$ are fixed points, i.e.*

$$\omega_w(x_n) \subset F.$$

Proof. To prove claim (1) is enough to note that:

$$\begin{aligned} \|x_n - W_n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|x_n - W_n x_n\| \\ &\quad +(1-\alpha_n)\mu_n\|DW_n x_n\| + \alpha_n\beta_n\|(I-S)x_n\| \Rightarrow \\ (1-\alpha_n)\|x_n - W_n x_n\| &\leq \|x_n - x_{n+1}\| + \mu_n\|DW_n x_n\| + \alpha_n\beta_n\|(I-S)x_n\| \end{aligned} \tag{3.3}$$

and the claim directly follows.
Moreover, observe that

$$\omega_s(x_n) = \omega_s(W_n x_n) \text{ and } \omega_w(x_n) = \omega_w(W_n x_n).$$

Suppose that (2) does not hold. As a consequence, let $p_0 \in \omega_w(x_n) \setminus F$ and denote by $(x_{n_k})_{k\in\mathbb{N}}$ a subsequence of (x_n) such that $x_{n_k} \rightharpoonup p_0$. By using (3.3), by the asymptotical regularity and by the Opial's property of a Hilbert space:

$$\begin{aligned} \liminf_{k\to\infty} \|x_{n_k} - p_0\| &< \liminf_{k\to\infty} \|x_{n_k} - Wp_0\| \\ &\leq \liminf_{k\to\infty} [\|x_{n_k} - W_{n_k}x_{n_k}\| + \|W_{n_k}x_{n_k} - W_{n_k}p_0\| \\ &\quad +\|W_{n_k}p_0 - Wp_0\|] \\ &\leq \liminf_{k\to\infty} [\|x_{n_k} - W_{n_k}x_{n_k}\| + \|x_{n_k} - p_0\| + \|W_{n_k}p_0 - Wp_0\|] \\ &\leq \liminf_{k\to\infty} \|x_{n_k} - p_0\| \end{aligned}$$

which is a contradiction. Then $p_0 \in F$ as required. □

Theorem 3.3. *Suppose that*

- (H2) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n\in\mathbb{N}} \mu_n = \infty$ *and* $\lim_{n\to\infty} \frac{|\mu_{n-1} - \mu_n|}{\mu_n} = 0$;
- (H3) $\lim_{n\to\infty} \frac{\sup_{z\in B}\|W_n z - W_{n-1}z\|}{\mu_n} = 0$, *with $B \subset H$ bounded.*
- (H4) $\lim_{n\to\infty} \frac{|\alpha_{n-1} - \alpha_n|}{\mu_n} = 0$.

Moreover, suppose that

$$\tau = 0 \tag{H1*}$$

holds.
Then the sequence generated by $x_1 \in H$ and the iteration (3.1) strongly converges to $x^* \in F$ that is the unique solution of the variational inequality

$$\langle Dx^*, y - x^*\rangle \geq 0, \qquad \forall y \in F. \tag{3.4}$$

Remark 3.4. The choice:

$$\alpha_n = \mu_n = \beta_n = \frac{1}{\sqrt{n}}, \ n \geq 1$$

satisfies our hypotheses.
Moreover, let A be a δ-inverse strongly monotone operator with $A^{-1}0 \neq \emptyset$ (see [29]) and let $(\lambda_n)_{n\in\mathbb{N}} \subset (0, 2\delta]$ be a sequence converging to $\lambda \in (0, 2\delta]$.
We have already stated that $W_n := (I - \lambda_n A)$ is a family of nonexpansive mappings with common fixed points $F = A^{-1}0$ satisfying (h1) and (h2) (see page 419 in [29]).
In this particular case, case (H3) of Theorem 3.3 reduces to

$$\lim_{n\to\infty} n|\lambda_{n-1} - \lambda_n| = 0.$$

To prove our first converge result we use the following proved in [20] as Theorem 2.2.

Lemma 3.5. *Let $(W_n)_{n\in\mathbb{N}}$ a sequence of nonexpansive mappings defined on H with common fixed points set $F \neq \emptyset$ satisfying (h1) and (h2).*
Let $D : H \to H$ be a $\sigma-$strongly monotone and $L-$lipschitzian operator.
Let us choose $(\mu_n)_{n\in\mathbb{N}} \subset (0, \mu)$ with $\mu < \dfrac{2\sigma}{L^2}$ such that (H2) and (H3) hold.
Let us choose $(\alpha_n)_{n\in\mathbb{N}} \subset (0, \alpha] \subset (0, 1)$ such that (H4) holds.
Then the sequence generated by $x_0 \in H$ and the iteration

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(I - \mu_n D)W_n x_n$$

strongly converges to $x^* \in F$ that is the unique solution of the variational inequality

$$\langle Dx^*, y - x^*\rangle \geq 0, \qquad \forall y \in F \tag{3.5}$$

Proof of Theorem 3.3. Fix $x_1 \in H$ and consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n := \|x_n - z_n\|$ and $(z_n)_{n\in\mathbb{N}}$ is generated by the iteration

$$\begin{cases} z_1 = x_1; \\ z_{n+1} = \alpha_n z_n + (1 - \alpha_n)(I - \mu_n D)W_n z_n, \quad n \geq 1. \end{cases} \tag{3.6}$$

Note that by Theorem 2.2 in [20], $(z_n)_{n\in\mathbb{N}}$ strongly converges to the unique solution x^* of the VIP (3.4).
To complete the proof, we have only to prove that a_n converges to 0. To this end, we compute

$$\begin{aligned} a_{n+1} &= \|x_{n+1} - z_{n+1}\| \\ &\leq \alpha_n\|x_n - z_n\| + (1 - \alpha_n)\|B_n W_n x_n - B_n W_n z_n\| + \alpha_n\beta_n\|Sx_n - x_n\| \\ &\leq (1 - (1 - \alpha_n)\mu_n\rho)\|x_n - z_n\| + \alpha_n\beta_n\|Sx_n - x_n\| \\ &= ((1 - (1 - \alpha_n)\mu_n\rho)a_n + \alpha_n\beta_n\|Sx_n - x_n\| \\ &= (1 - \gamma_n)a_n + \alpha_n\beta_n\|Sx_n - x_n\|, \end{aligned}$$

where $\gamma_n := (1 - \alpha_n)\mu_n\rho$. Note that $\|Sx_n - x_n\|$, since $(x_n)_{n\in\mathbb{N}}$ is bounded by Lemma 3.1 and S is nonexpansive. Hence, it holds

$$\lim_n \frac{\alpha_n\beta_n\|Sx_n - x_n\|}{\gamma_n} = 0$$

by (H1*).
By this last an (H2), we can apply Lemma 2.5 to obtain that $a_n \to 0$ and

$$\lim_n \|x_n - x^*\| = \lim_n a_n + \|z_n - x^*\| = 0.$$

$\square$

Theorem 3.6. *Suppose that*

(H2) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n\in\mathbb{N}} \mu_n = \infty$ *and* $\lim_{n\to\infty} \frac{|\mu_n - \mu_{n-1}|}{\alpha_n\beta_n\mu_n} = 0$;

(H5) $\lim_{n\to\infty} \frac{\sup_{z\in B} \|W_n z - W_{n-1}z\|}{\alpha_n\beta_n\mu_n} = 0$, *with* $B \subset H$ *bounded.*

(H6) $\lim_{n\to\infty} \frac{|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n\mu_n} = 0$;

(H7) *there exists* $K > 0$ *such that* $\left| \frac{1}{\alpha_n\beta_n} - \frac{1}{\alpha_{n-1}\beta_{n-1}} \right| \le K\mu_n$.

Moreover, assume that

$$\tau \in (0, +\infty). \tag{H1**}$$

Then $x_n \to \tilde{x}$, *as* $n \to \infty$, *where* $\tilde{x} \in F$ *is the unique solution of the variational inequality*

$$\langle \frac{1}{\tau} D\tilde{x} + (I - S)\tilde{x}, y - \tilde{x}\rangle \ge 0, \qquad \forall y \in F. \tag{3.7}$$

Remark 3.7. We note that conditions (H2)-(H7) cannot be satisfied by the same sequence of Remark 3.4, since

$$\lim_{n\to\infty} \frac{1}{\mu_n} \left| \frac{1}{\alpha_n\beta_n} - \frac{1}{\alpha_{n-1}\beta_{n-1}} \right| = \lim_{n\to\infty} \sqrt{n} = \infty.$$

However a simple calculation shows that

$$\mu_n = \frac{2}{\sqrt{n}} \text{ and } \alpha_n = \beta_n = \frac{1}{\sqrt[4]{n}}, n \ge 1$$

satisfy the hypotheses of Theorem 3.6.

Proof. At first, we point out that the problem is well-posed. Indeed, Lemma 2.3 ensures that $\left(\frac{1}{\tau}D + (I - S)\right)$ is a strongly monotone operator, so that (3.7) has a unique solution $\tilde{x}$.
Moreover, observe that $(x_n)_{n\in\mathbb{N}}$ is bounded by (H1**) and Lemma 3.1.
We compute

$$\begin{aligned}
x_n - x_{n+1} &= (1 - \alpha_n)(x_n - B_n W_n x_n) + \alpha_n\beta_n(I - S)x_n \\
&= (1 - \alpha_n)(x_n - W_n x_n + \mu_n D W_n x_n) + \alpha_n\beta_n(x_n - Sx_n) \\
&= (1 - \alpha_n)(I - W_n)x_n + (1 - \alpha_n)\mu_n D W_n x_n + \alpha_n\beta_n(I - S)x_n
\end{aligned}$$

and define

$$v_n := \frac{x_n - x_{n+1}}{\alpha_n\beta_n} = (I - S)x_n + \frac{1 - \alpha_n}{\alpha_n\beta_n}(I - W_n)x_n + \frac{(1 - \alpha_n)\mu_n}{\alpha_n\beta_n} D W_n x_n. \tag{3.8}$$

We will prove that $v_n \to 0$ as $n \to \infty$; this means that $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular with respect to $(\alpha_n\beta_n)_{n\in\mathbb{N}}$, i.e.

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\alpha_n\beta_n} = 0.$$

To do so, we compute

$$\begin{aligned} x_{n+1}-x_n &= \alpha_n(x_n-x_{n-1})+(\alpha_n-\alpha_{n-1})(x_{n-1}-B_{n-1}W_{n-1}x_{n-1}) \\ &\quad +(1-\alpha_n)(B_nW_ny_n-B_{n-1}W_{n-1}y_{n-1}) \\ &\quad +\alpha_n\beta_n(Sx_n-x_n)-\alpha_{n-1}\beta_{n-1}(Sx_{n-1}-x_{n-1}) \end{aligned} \tag{3.9}$$

and note that

$$\begin{aligned} &\alpha_n\beta_n(Sx_n-x_n)-\alpha_{n-1}\beta_{n-1}(Sx_{n-1}-x_{n-1}) \\ &=\alpha_n\beta_nSx_n-\alpha_n\beta_nx_n-\alpha_{n-1}\beta_{n-1}Sx_{n-1}+\alpha_{n-1}\beta_{n-1}x_{n-1}) \\ &=\alpha_n\beta_n(Sx_n-Sx_{n-1})+(\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1})Sx_{n-1} \\ &\quad -\alpha_n\beta_n(x_n-x_{n-1})-(\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1})x_{n-1} \\ &=\alpha_n\beta_n(Sx_n-Sx_{n-1})+(\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1})(Sx_{n-1}-x_{n-1}) \\ &\quad -\alpha_n\beta_n(x_n-x_{n-1}) \end{aligned} \tag{3.10}$$

By means of (3.10), computation (3.9) becomes

$$\begin{aligned} x_{n+1}-x_n &= \alpha_n(x_n-x_{n-1})+(\alpha_n-\alpha_{n-1})(x_{n-1}-B_{n-1}W_{n-1}x_{n-1}) \\ &\quad +(1-\alpha_n)(B_nW_ny_n-B_{n-1}W_{n-1}y_{n-1}) \\ &\quad +\alpha_n\beta_n(Sx_n-Sx_{n-1})+(\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1})(Sx_{n-1}-x_{n-1}) \\ &\quad -\alpha_n\beta_n(x_n-x_{n-1}) \\ &= \alpha_n(1-\beta_n)(x_n-x_{n-1})+(\alpha_n-\alpha_{n-1})(x_{n-1}-B_{n-1}W_{n-1}x_{n-1}) \\ &\quad +(1-\alpha_n)(B_nW_ny_n-B_{n-1}W_{n-1}y_{n-1}) \\ &\quad +\alpha_n\beta_n(Sx_n-Sx_{n-1})+(\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1})(Sx_{n-1}-x_{n-1}), \end{aligned}$$

so, passing to the norm and using the nonexpansivity of S

$$\begin{aligned} \|x_{n+1}-x_n\| &\leq \alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|x_{n-1}-B_{n-1}W_{n-1}x_{n-1}\| \\ &\quad +(1-\alpha_n)\|B_nW_ny_n-B_{n-1}W_{n-1}y_{n-1}\| \\ &\quad +|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|\|Sx_{n-1}-x_{n-1}\|. \end{aligned} \tag{3.11}$$

Moreover

$$\begin{aligned} \|B_nW_nx_n-B_{n-1}W_{n-1}x_{n-1}\| &\leq \|B_nW_nx_n-B_nW_{n-1}x_{n-1}\| \\ &\quad +\|B_nW_{n-1}x_{n-1}-B_{n-1}W_{n-1}x_{n-1}\| \\ &\leq (1-\mu_n\rho)\|W_nx_n-W_{n-1}x_{n-1}\| \\ &\quad +\|W_{n-1}x_{n-1}-\mu_nDW_{n-1}x_{n-1} \\ &\qquad -W_{n-1}x_{n-1}+\mu_{n-1}DW_{n-1}x_{n-1}\| \\ &\leq (1-\mu_n\rho)\|x_n-x_{n-1}\|+\|W_nx_{n-1}-W_{n-1}x_{n-1}\| \\ &\quad +|\mu_n-\mu_{n-1}|\|DW_{n-1}x_{n-1}\| \end{aligned} \tag{3.12}$$

By means of (3.12), in (3.11) we have that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \alpha_n\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|x_{n-1}-B_{n-1}W_{n-1}x_{n-1}\|\\ &\quad +(1-\alpha_n)(1-\mu_n\rho)\|x_n-x_{n-1}\|+(1-\alpha_n)\|W_nx_{n-1}-W_{n-1}x_{n-1}\|\\ &\quad +(1-\alpha_n)|\mu_n-\mu_{n-1}|\|DW_{n-1}x_{n-1}\|\\ &\quad +|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|\|Sx_{n-1}-x_{n-1}\|\\ &= (1-(1-\alpha_n)\mu_n\rho)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|x_{n-1}-B_{n-1}W_{n-1}x_{n-1}\|\\ &\quad +\|W_nx_{n-1}-W_{n-1}x_{n-1}\|+|\mu_n-\mu_{n-1}|\|DW_{n-1}x_{n-1}\|\\ &\quad +|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|\|Sx_{n-1}-x_{n-1}\|.\end{aligned}$$

By this last and by the boundedness of $(x_n)_{n\in\mathbb{N}}$, there exists $M>0$ such that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-(1-\alpha_n)\mu_n\rho)\|x_n-x_{n-1}\|+\|W_nx_{n-1}-W_{n-1}x_{n-1}\|\\ &\quad (\||\mu_n-\mu_{n-1}|+|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|+|\alpha_n-\alpha_{n-1}|)M. \quad (3.13)\end{aligned}$$

Moreover, by (H5)

$$\lim_{n\to\infty}\frac{\|W_nx_{n-1}-W_{n-1}x_{n-1}\|}{\mu_n}=0$$

and by (H6)

$$\lim_{n\to\infty}\frac{|\mu_n-\mu_{n-1}|+|\alpha_n-\alpha_{n-1}|+|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\mu_n}=0.$$

We can apply Lemma 2.5 to obtain that $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular.
Dividing by $\alpha_n\beta_n$ in (3.13), it can be observed that

$$\begin{aligned}\frac{\|x_{n+1}-x_n\|}{\alpha_n\beta_n} &\leq (1-(1-\alpha_n)\mu_n\rho)\frac{\|x_n-x_{n-1}\|}{\alpha_n\beta_n}+\frac{\|W_nx_{n-1}-W_{n-1}x_{n-1}\|}{\alpha_n\beta_n}\\ &\quad +\frac{|\mu_n-\mu_{n-1}|+|\alpha_n-\alpha_{n-1}|+|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n}M\\ &\leq (1-(1-\alpha_n)\mu_n\rho)\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}\beta_{n-1}}+\|x_{n-1}-x_n\|\left|\frac{1}{\alpha_n\beta_n}-\frac{1}{\alpha_{n-1}\beta_{n-1}}\right|\\ &\quad +\frac{\|W_nx_{n-1}-W_{n-1}x_{n-1}\|}{\alpha_n\beta_n}\\ &\quad +\frac{|\mu_n-\mu_{n-1}|+|\alpha_n-\alpha_{n-1}|+|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n}M\\ \text{by (H7)} &\leq (1-(1-\alpha_n)\mu_n\rho)\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}\beta_{n-1}}+\mu_nK\|x_{n-1}-x_n\|\\ &\quad +\frac{\|W_nx_{n-1}-W_{n-1}x_{n-1}\|}{\alpha_n\beta_n}\\ &\quad +\frac{|\mu_n-\mu_{n-1}|+|\alpha_n-\alpha_{n-1}|+|\alpha_n\beta_n-\alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n}M\end{aligned}$$

Since (H2), (H5) and (H6) hold, we can apply again Lemma 2.5 to obtain that

$$\lim_{n\to\infty}v_n=\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n\beta_n}=0.$$

Now, fix $z \in F$; by (3.8) it results

$$\begin{aligned}\langle v_n, x_n - z\rangle &= \langle (I-S)x_n, x_n - z\rangle + \frac{1-\alpha_n}{\alpha_n\beta_n}\langle (I-W_n)x_n, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle DW_nx_n, x_n - z\rangle \\ &= \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &\quad + \frac{1-\alpha_n}{\alpha_n\beta_n}\langle (I-W_n)x_n - (I-W_n)z, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle DW_nx_n - Dx_n, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle Dx_n - Dz, x_n - z\rangle + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle Dz, x_n - z\rangle\end{aligned}$$

By using the strong monotonicity of D and the monotonicity of $(I-S)$ we have

$$\begin{aligned}\langle v_n, x_n - z\rangle &\geq \langle (I-S)z, x_n - z\rangle + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle Dz, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle DW_nx_n - Dx_n, x_n - z\rangle + \frac{(1-\alpha_n)\mu_n\sigma}{\alpha_n\beta_n}\|x_n - z\|^2\end{aligned} \tag{3.14}$$

Since D is Lipschitzian and by claim (1) of Lemma 3.2 we easily show that both $(\|DW_nx_n - Dx_n\|)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are null sequences.
Passing to the limit (3.14), we obtain that all weak cluster points of $(x_n)_{n\in\mathbb{N}}$ are strong cluster points, i.e.

$$\omega_w(x_n) = \omega_s(x_n)$$

and in light of claim (1) of Lemma 3.2

$$\omega_w(x_n) = \omega_s(x_n) = \omega_w(W_nx_n) = \omega_s(W_nx_n).$$

Note that $\omega_w(x_n)$ is not empty since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in a Hilbert space. To end the proof, let $x' \in \omega_w(x_n)$ and let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ converging to x'. Then, by (3.8),

$$\langle v_{n_k}, x_{n_k} - z\rangle \geq \langle (I-S)x_{n_k}, x_{n_k} - z\rangle + \frac{(1-\alpha_{n_k})\mu_{n_k}}{\alpha_{n_k}\beta_{n_k}}\langle DW_nx_{n_k}, x_{n_k} - z\rangle$$

For any $z \in F$.
Passing to $k \to \infty$ we obtain

$$\frac{1}{\tau}\langle Dx', x' - z\rangle + \langle (I-S)z, x' - z\rangle \leq 0 \qquad \forall z \in F$$

which coincides with (3.7). From the uniqueness of the solution of this last, we deduce that $x' = \tilde{x}$, i.e. $\omega_w(x_n) = \omega_s(x_n) = \{\tilde{x}\}$ and this, of course, ensures that $x_n \to \tilde{x}$, as $n \to \infty$. □

It remains to investigate the case

$$\tau = +\infty. \tag{H1***}$$

Theorem 3.8. *Assume that* (H1***) *holds. Moreover, suppose that*

(H2) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n\in\mathbb{N}} \mu_n = \infty$ *and* $\lim_{n\to\infty} \frac{|\mu_n - \mu_{n-1}|}{\alpha_n\beta_n\mu_n} = 0$*;*

(H5) $\displaystyle\lim_{n\to\infty} \frac{\sup_{z\in B}\|W_n z - W_{n-1}z\|}{\alpha_n\beta_n\mu_n} = 0$, *with* $B \subset H$ *bounded;*

(H6) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\displaystyle\lim_{n\to\infty}\frac{|\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|}{\alpha_n\beta_n\mu_n} = 0$;

(H7) *there exists* $K > 0$ *such that* $\displaystyle\frac{1}{\alpha_n\mu_n}\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \le K$.

If $(x_n)_{n\in\mathbb{N}}$ *is bounded, then every* $v \in \omega_w(x_n)$ *is solution of the variational inequality*

$$\langle (I-S)v, v-x\rangle \le 0, \qquad \forall x \in F.$$

Proof. Following Theorem 3.6, the boundedness of $(x_n)_{n\in\mathbb{N}}$ permits to prove that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n\beta_n} = 0.$$

From this last and by applying Lemma 3.2, we obtain that $\omega_w(x_n) \subset F$. By using the same notations adopted in the proof of Theorem 3.6 and by (3.14), we have that

$$\begin{aligned}\langle v_n, x_n - z\rangle \quad &\ge \quad \langle (I-S)z, x_n - z\rangle + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle Dz, x_n - z\rangle \\ &\quad + \frac{(1-\alpha_n)\mu_n}{\alpha_n\beta_n}\langle DW_n x_n - Dx_n, x_n - z\rangle\end{aligned}$$

holds for all $z \in F$. Thus let $v \in \omega_w(x_n)$, $x_{n_k} \rightharpoonup v \in F$ and note that by (H1***), $\displaystyle\lim_{k\to\infty}\frac{\mu_{n_k}}{\alpha_{n_k}\beta_{n_k}} = 0$. Moreover, $(x_n)_{n\in\mathbb{N}}$ is bounded , $v_n \to 0$ and $\|(DW_n x_n - Dx_n)\| \to 0$ as $n \to \infty$. Then, we have

$$\begin{aligned}\langle (I-S)z, v-z\rangle \quad &= \quad \lim_k \langle (I-S)z, x_{n_k} - z\rangle \\ &\le \quad \lim_k [\langle v_{n_k}, x_{n_k} - z\rangle + \frac{(1-\alpha_{n_k})\mu_{n_k}}{\alpha_{n_k}\beta_{n_k}}\langle Dz, x_{n_k} - z\rangle \\ &\quad + \frac{(1-\alpha_{n_k})\mu_{n_k}}{\alpha_{n_k}\beta_{n_k}}\langle DW_{n_k}x_{n_k} - Dx_{n_k}, x_{n_k} - z\rangle] \\ &\le \quad 0\end{aligned}$$

for any $z \in F$.

Since F is the fixed point set of a nonexpansive mappings by assumption (h2), we can substitute z with $v + \mu(z-v)$, $\mu \in (0,1)$ to obtain

$$\langle (I-S)(v + \mu(z-v)), v - z\rangle \le 0.$$

Finally, letting $\mu \to 0$,

$$\langle (I-S)v, v-z\rangle \le 0, \qquad \forall z \in F.$$

□

Remark 3.9. An example of sequences satisfying the above conditions is given by $\alpha_n = \beta_n = \mu_n = \dfrac{1}{\sqrt[4]{n}}$.

We note that the boundedness requirement on $(x_n)_{n\in\mathbb{N}}$ is justified by the next example.

Example 3.10. *Let $H = \mathbb{R}$, $x_0 = 1$, $W_n x := x$ for any $n \in \mathbb{N}$, $Sx = x + 1$, $\alpha_n = \frac{1}{\sqrt{n}}$, $\beta_n = 1$, $\mu_n = \frac{1}{n}$ and $B_n(x) = \left(1 - \frac{1}{n}\right) x$. Our method becomes:*

$$x_{n+1} = \frac{1}{\sqrt{n}} x_n + \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{\sqrt{n}}\right) x_n + \frac{1}{\sqrt{n}}$$

and the sequence is not bounded. Indeed, suppose the contrary, i.e. that there exists $M > 0$ with the property that $|x_n| \leq M$. Then

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{\sqrt{n}} - \left(1 - \frac{1}{\sqrt{n}}\right)\frac{1}{n} x_n \geq \left(\frac{1}{\sqrt{n}} - \frac{M}{n}\right) + \frac{1}{\sqrt{n^3}} \\ &\geq \left(\frac{1}{\sqrt{n}} - \frac{M}{n}\right) \geq \frac{1}{2\sqrt{n}} \end{aligned}$$

for any $n \geq 4M^2$, which contradicts the divergence of $\sum_{n=1}^{\infty} n^{-1/2}$.

Lemma 3.1 states that if S has a fixed point in F, then the sequence (x_n) is bounded. In this case, Problem (1.7) could be ill-posed because every fixed point is a solution. We will show that in this situation, the operator D acts as a regularization.

Theorem 3.11. *Suppose that:*

(H2) $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n\in\mathbb{N}} \mu_n = \infty$ *and* $\lim_{n\to\infty} \frac{|\mu_{n-1} - \mu_n|}{\mu_n} = 0$;

(H3) $\lim_{n\to\infty} \frac{\sup_{z\in B} \|W_n z - W_{n-1} z\|}{\mu_n} = 0$, *with* $B \subset H$ *bounded.*

(H4) $\lim_{n\to\infty} \frac{|\alpha_{n-1} - \alpha_n|}{\mu_n} = 0$.

(H8) $\lim_{n\to\infty} \frac{|\alpha_{n-1}\beta_{n-1} - \alpha_n\beta_n|}{\mu_n} = 0$.

Morover, assume that $\lim_{n\to\infty} \alpha_n\beta_n = \theta \neq 0$.
If the set $F \cap Fix(S)$ is not empty, then the sequence generated by $x_1 \in H$ and the iteration (3.1) strongly converges to $x^* \in F \cap Fix(S)$ that is the unique solution of the variational inequality

$$\langle Dx^*, y - x^* \rangle \geq 0, \qquad \forall y \in F \cap Fix(S). \tag{3.15}$$

Proof. By Lemma 3.1, since $Fix(S) \cap F \neq \emptyset$, we know that $(x_n)_{n\in\mathbb{N}}$ is bounded by some constant $M > 0$. By using (3.13) in Theorem 3.6 we have

$$\begin{aligned} \|x_{n+1} - x_n\| \leq\ & (1 - (1 - \alpha_n)\mu_n\rho)\|x_n - x_{n-1}\| + \|W_n x_{n-1} - W_{n-1} x_{n-1}\| \\ & (\||\mu_n - \mu_{n-1}| + |\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)M \end{aligned} \tag{3.16}$$

Let us observe that by (H2), (H4) and (H8),

$$\lim_{n\to\infty} \frac{|\mu_n - \mu_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\alpha_n\beta_n - \alpha_{n-1}\beta_{n-1}|}{\mu_n} = 0;$$

moreover, (H3) guarantees that

$$\lim_{n\to\infty} \frac{\|W_n y_{n-1} - W_{n-1} y_{n-1}\|}{\mu_n} = 0.$$

Hence, we can apply Lemma 2.5 to ensure that $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular.
Fix $v \in Fix(S) \cap F$, then:

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\leq \alpha_n\|\beta_n(Sx_n - v) + (1-\beta_n)(x_n - v)\|^2 + (1-\alpha_n)\|B_nW_nx_n - v\|^2 \\
&\leq \alpha_n\beta_n\|Sx_n - v\|^2 + \alpha_n(1-\beta_n)\|x_n - v\|^2 - \alpha_n\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 \\
&\quad +(1-\alpha_n)\|(W_nx_n - v) - \mu_nDW_nx_n\|^2 \\
&\leq \alpha_n\|x_n - v\|^2 - \alpha_n\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 + (1-\alpha_n)\|W_nx_n - v\|^2 \\
&\quad +\mu_n^2\|DW_nx_n\|^2 - 2(1-\alpha_n)\mu_n\langle W_nx_n - v, DW_nx_n\rangle. \qquad (3.17)
\end{aligned}$$

By the boundedness of (x_n), there exists $L > 0$ such that:

$$\begin{aligned}
\alpha_n\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 &\leq +\mu_n^2\|DW_nx_n\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
&\quad -2(1-\alpha_n)\mu_n\langle W_nx_n - v, DW_nx_n\rangle \\
&\leq (\mu_n + \|x_n - x_{n+1}\|)M \qquad (3.18)
\end{aligned}$$

Then $\|Sx_n - x_n\| \to 0$ as $n \to \infty$ and, by demiclosedness principle, the weak cluster points of $(x_n)_{n\in\mathbb{N}}$ are fixed points of S, i.e. $\omega_w(x_n) \subset Fix(S)$.
Moreover, following (3.3), since $\alpha_n\beta_n \to \theta \neq 0$ and $\|x_n - Sx_n\| \to 0$ then $\|x_n - W_nx_n\| \to 0$ and, as a consequence, $\|x_{n+1} - W_nx_n\| \to 0$.

Let us show that $\omega_w(x_n) \subset F$. Assume the contrary, i.e. suppose that there exists $p_0 \in \omega_w(x_n)$ and $p_0 \notin F$. By Opial's property of Hilbert space:

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k} - p_0\| &< \liminf_{k\to\infty}\|x_{n_k} - Wp_0\| \\
&\leq \liminf_{k\to\infty}[\|x_{n_k} - x_{n_k+1}\| + \|x_{n_k+1} - W_{n_k}x_{n_k}\| \\
&\quad +\|W_{n_k}x_{n_k} - W_{n_k}p_0\| + \|W_{n_k}p_0 - Wp_0\|] \\
&\leq \liminf_{k\to\infty}[\|x_{n_k} - x_{n_k+1}\| + \|x_{n_k+1} - W_{n_k}x_{n_k}\| \\
&\quad +\|x_{n_k} - p_0\| + \|W_{n_k}p_0 - Wp_0\|] \leq \liminf_{k\to\infty}\|x_{n_k} - p_0\|
\end{aligned}$$

that is a contradiction, so $p_0 \in F$.
To conclude, if z is the unique solution of VIP (3.15),

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n(x_n - z) + (1-\alpha_n)(B_nW_nx_n - z) + \alpha_n\beta_n(S - I)x_n\|^2 \\
&= \|\alpha_n(x_n - z) + (1-\alpha_n)(B_nW_nx_n - B_nz) + (1-\alpha_n)(B_nz - z) \\
&\qquad +\alpha_n\beta_n(S-I)x_n\|^2 \\
&\leq \|\alpha_n(x_n - z) + (1-\alpha_n)(B_nW_nx_n - B_nz) + \alpha_n\beta_n(Sx_n \pm z - x_n)\|^2 \\
&\quad +2(1-\alpha_n)\mu_n\langle -Dz, x_{n+1} - z\rangle \\
&\leq (1-\alpha_n)(1-\mu_n\rho)\|x_n - z\|^2 + \alpha_n\|x_n - z\|^2 \\
&\quad +2(1-\alpha_n)\mu_n\langle -Dz, x_{n+1} - z\rangle \\
&= (1-(1-\alpha_n)\mu_n\rho)\|x_n - z\|^2 + 2\alpha_n\langle -Dz, x_{n+1} - z\rangle.
\end{aligned}$$

Since every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ is in $F \cap Fix(S)$, let (x_{n_k}) be such that $x_{n_k} \rightharpoonup p_0$. Then, by passing to a further subsequence, if necessary, we have

$$\limsup_{n\to\infty}\langle -Dz, x_{n+1} - z\rangle = \lim_{k\to\infty}\langle -Dz, x_{n_k} - z\rangle = \langle -Dz, p_0 - z\rangle \leq 0.$$

Applying Lemma 2.5, we get $x_n \to z$ as $n \to \infty$. □

We stress that we can still recover the result of the previous theorem by slightly changing the assumptions on the sequences, as the next result shows.

Theorem 3.12. *Suppose that (H2), (H5), (H6) and (H7).*
Moreover, assume that $\lim_{n\to\infty} \alpha_n\beta_n = 0$ *and that* $F \cap Fix(S)$ *is not empty.*
Then the sequence generated by $x_0 \in H$ *and the iteration (3.1) strongly converges to* $x^* \in F \cap Fix(S)$ *that is the unique solution of the variational inequality*

$$\langle Dx^*, y - x^* \rangle \geq 0, \qquad \forall y \in F \cap Fix(S).$$

Proof. The proof follows as in the previous case for the boundedness and the asymptotical regularity.

By proceeding as in Theorem 3.6, we can prove that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\alpha_n\beta_n} = 0.$$

Then in (3.18) we can divide by $\alpha_n\beta_n$ and pass to the limit to obtain that $\|Sx_n - x_n\| \to 0$ as $n \to \infty$. The remaining part follows again as in the proof of the previous theorem. □

References

[1] S. Atsushiba S. Takahashi W., Strong convergence theorems for a finite family of nonexpansive mappings and applications, Indian J. Math., 41 3 (1999) 435-453.
[2] H.H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl., 202 (1996), pp. 150159.
[3] Bauschke H. H., Borwein J. M., On projection algorithms for solving convex feasibility problem, SIAM Review 38(1966), 367-426.
[4] Bauschke, H. H., Borwein J. M., On the convergence of von Neumann's alternating projection algorithm for two sets. Set-Valued Analysis 1.2(1993), 185-212.
[5] Byrne C., A unified treatment of some iterative algorithms in signal processing and image reconstrucion, Inverse Problems 20 (2004), 103-120
[6] Browder F. E., and Petryshyn V. W., Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 2 (1967), 197–228.
[7] Bruck R. E., Properties of fixed point sets of nonexapnsive mapping in Banach spaces, Transactions of the Amer. Math. Soc. 179 (1973), 251–262.
[8] Cabot A., Proximal point algorithm controlled by a slowly vanishing term: applications to hierarchical minimization, SIAM J. Optim. 15 (2005), 555-572
[9] Cianciaruso F., Marino G., Muglia L., Yao Y., On a two-step algorithm for hierarchical fixed point problems and variational inequalities, Journal of Inequalities and Applications, Article ID 208692, 13 pages doi:10.1155/2009/208692
[10] Colao V., Marino G., Common fixed points of strict pseudocontractions by iterative algorithms, J. Math. Anal. Appl., 382 2 (2011), 631-644.
[11] Colao V., Marino G., Muglia L.. Viscosity methods for common solutions for equilibrium and hierarchical fixed point problems, Optimization 60.5 (2011), 553–573.
[12] Colao V., Marino G., Muglia L., On some auxiliary mappings generated by nonexpansive and strictly pseudo-contractive mappings, Applied Math Comp. 218 11 (2012), 6232–6241.
[13] Combettes, P. L.; Hirstoaga, S. A., Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. 6 (2005), no. 1, 117-136.
[14] Deimling K., Nonlinear functional analysis, Courier Dover Publications, 2013.
[15] Deutsch, F.; Yamada, I.. Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings. Numer. Funct. Anal. Optim. 19 (1998), no. 1-2, 33–56.
[16] Goebel, K.; Kirk, W. A., Topics in metric fixed point theory. Cambridge Studies in Advanced Mathematics, 28. Cambridge University Press, Cambridge, 1990.
[17] Gu G., Wang S., Cho Y.J., Strong convergence algorithms for hierarchical fixed points problems and variational inequalities, Journal of Applied Mathematics 2011 (2011).
[18] Hirstoaga, S. A., Iterative selection methods for common fixed point problems, JJ. Math. Anal. Appl., 324 2 (2006), 1020-1035.

[19] Marino G., Muglia L., Yao Y., Viscosity methods for common solutions of equilibrium and variational inequality problems via multi-step iterative algorithms and common fixed points, Nonlinear Analysis: TMA 75 4 (2012), 1787–1798.
[20] Marino G., Muglia, L., On the auxiliary mappings generated by a family of mappings and solutions of variational inequalities problems, Optimization Lett. 2013 1–20
[21] Marino G., Muglia, L., Yao Y., The uniform asymptotical regularity of families of mappings and solutions of variational inequality problems, J. of Nonlinaer Convex Anal. 15 (3) (2014), 477–492
[22] Marino, G.; Xu, H. K., A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. 318 (2006), no. 1, 43-52.
[23] Marino, G.; Xu, H. K., Explicit hierarchical fixed point approach to variational inequalities, J. Optimization Theory and Appl. 149 (2011) 61–78.
[24] Mainge P.E.; Moudafi, A., Strong convergence of an iterative method for hierarchical fixed point problems, Pacific J. Optim., 3 (2007), 529-538.
[25] Moudafi, A., Krasnoselski-Mann iteration for hierarchical fixed-point problems, Inverse Problems, 23 (2007), 1635-1640.
[26] Reich, S., Xu H.K., An iterative approach to a constrained least squares problem, Abstr. Appl. Anal. 8 (2003), 503–512.
[27] Solodov M., An explicit descent method for bilevel convex optimization, J. Convex Anal. 14 (2007), 227-237
[28] Shimoji K., Takahashi W., Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese J. Math 5 (2001) 387–404.
[29] Takahashi W., Toyoda M. Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003), no. 2, 417–428.
[30] Xu, H.K., Iterative Algorithms for Nonlinear Operators, J. London Math. Soc. 2 (2002), 1-17.
[31] Xu H.K., Kim T.H., Convergence of hybrid steepest-descent methods for variational inequalities, J. Optim. Theory Appl. 119 (2003), no. 1, 185–201.
[32] Xu, H. K., Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., 298 (2004), 279-291
[33] Yamada I., The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings *Inheretly Parallel Alg. in Feasibility and Opt. and their Appl.* (2001) 473-504

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Dear Professor Colao and Muglia,

I am delighted to let you know that your paper entitled
"A hierarchical approach to fixed point problems for uniformly asymptotically regular sequences"
Has been accepted for publication in the special issue of the Journal of Nonlinear and Convex Analysis (JNCA) that is dedicated to the 70th birthday of Professor Wataru Takahashi. Please check your paper once again for every detail and then send me the tex file at your earliest convenience. Thank you once again for your contribution to this special issue of JNCA.

Sincerely yours,

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Krasnoselskii-Mann method for non-self mappings

Vittorio Colao[1] · **Giuseppe Marino**[1,2]

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Abstract Let H be a Hilbert space and let C be a closed, convex and nonempty subset of H. If $T : C \to H$ is a non-self and non-expansive mapping, we can define a map $h : C \to \mathbb{R}$ by $h(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in C\}$. Then, for a fixed $x_0 \in C$ and for $\alpha_0 := \max\{1/2, h(x_0)\}$, we define the Krasnoselskii-Mann algorithm $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$, where $\alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}$. We will prove both weak and strong convergence results when C is a strictly convex set and T is an inward mapping.

1 Introduction

Let C be a closed, convex and nonempty subset of a Hilbert space H and let $T : C \to H$ be a non-expansive mapping, such that the fixed point set $Fix(T) := \{x \in C : Tx = x\}$ is not empty.

For a real sequence $\{\alpha_n\} \subset (0, 1)$, we will consider the iterations

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n. \end{cases} \tag{1}$$

If T is a self-mapping, the iterative scheme above had been studied in an impressive amount of papers (see the book [1] and references therein) in the last decades and it is often called "segmenting Mann" ([2], [3], [4]) or "Krasnoselskii-Mann" (e.g. [5],[6]) iteration.

A general result on the algorithm (1) is due to Reich [7] and states that the sequence $\{x_n\}$ weakly converges to a fixed point of the operator T under the following assumptions:

(C1) T is a self-mapping, i.e. $T : C \to C$ and

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(C2) $\{\alpha_n\}$ is such that $\sum_n \alpha_n(1-\alpha_n) = +\infty$.

In this paper, we are interested in lowering condition $(C1)$ by allowing T to be non-self, at the price of strengthening the requirements on the sequence $\{\alpha_n\}$ and on the set C.

Indeed, we will assume that C is a strictly convex set and that the non-expansive map $T : C \to H$ is inward.

Historically, the inward condition and its generalizations had been widely used to prove convergence results for both implicit ([8], [9], [10], [11]) and explicit (see, e.g., [12], [13], [1], [14]) algorithms.

However, we point out that the explicit case had been only studied in conjunction with processes involving the calculation of a projection or a retraction $P : H \to C$ at each step.

As an example, in [12], the following algorithm is studied:

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n),$$

where $T : C \to H$ satisfies the weakly inward condition, f is a contraction and $P : H \to C$ is a non-expansive retraction.

We point out that in many real world applications, the process of calculating P can be a resource consumption task and it may require an approximating algorithm by itself, even in the case when P is the nearest point projection.

To overcome the necessity of using an auxiliary mapping P, for an inward and non-expansive mapping $T : C \to H$, we will introduce a new search strategy for the coefficients $\{\alpha_n\}$ and we will prove that the Krasnoselskii-Mann algorithm

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n,$$

is well-defined for this particular choice of the sequence $\{\alpha_n\}$. Also we will prove both weak and strong convergence results for the above algorithm when C is a strictly convex set.

We stress that the main difference between the classical Krasnoselskii-Mann and our algorithm is that the choice of the coefficient α_n is not made a priori in the latter, but it is constructed step to step and determined by the values of the map T and the geometry of the set C.

2 Main Result

We will make use of the following

Definition 1 A map $T : C \to H$ is said to be inward (or to satisfy the inward condition) if, for any $x \in C$ it holds

$$Tx \in I_C(x) := \{x + c(u-x) : c \geq 1 \text{ and } u \in C\}. \tag{2}$$

We refer to [15] for a comprehensive survey on the properties of the inward mappings.

Definition 2 A set $C \subset H$ is said to be strictly convex if it is convex and with the property that $x, y \in \partial C$ and $t \in (0,1)$, implies that

$$tx + (1-t)y \in \mathring{C}.$$

In other words, if the boundary ∂C does not contain any segment.

Definition 3 A sequence $\{y_n\} \subset C$ is Fejér-monotone with respect a set $D \subset C$ if, for any element $y \in D$

$$\|y_{n+1} - y\| \le \|y_n - y\| \qquad \forall n \in \mathbb{N}.$$

For a closed and convex set C and a map $T : C \to H$, we define a mapping $h : C \to \mathbb{R}$ as

$$h(x) := \inf\{\lambda \ge 0 : \lambda x + (1-\lambda)Tx \in C\}. \tag{3}$$

Note that the above quantity in a minimum, since C is closed. In the following lemma, we group the properties of the function defined above.

Lemma 1 *Let C be a non-empty, closed and convex set, let $T : C \to H$ be a mapping and define $h : C \to \mathbb{R}$ as in (3). Then the following properties hold:*

(P1) for any $x \in C$, $h(x) \in [0,1]$ and $h(x) = 0$ if and only if $Tx \in C$;
(P2) for any $x \in C$ and any $\alpha \in [h(x), 1]$, $\alpha x + (1-\alpha)Tx \in C$;
(P3) if T is an inward mapping, then $h(x) < 1$ for any $x \in C$ and
(P4) whenever $Tx \notin C$, $h(x)x + (1-h(x))Tx \in \partial C$.

Proof Properties $(P1)$ and $(P2)$ follows directly from the definition of h. To prove $(P3)$, observe that (2) implies

$$\frac{1}{c}Tx + (1 - \frac{1}{c})x \in C$$

for some $c \ge 1$. As a consequence,

$$h(x) = \inf\{\lambda \ge 0 : \lambda x + (1-\lambda)Tx \in C\} \le (1 - \frac{1}{c}) < 1.$$

In order to verify $(P4)$, we first note that $h(x) > 0$ by property $(P1)$ and that $h(x)x + (1-h(x))Tx \in C$. Let $\{\eta_n\} \subset (0, h(x))$ be a sequence of real numbers converging to $h(x)$ and note that, by the definition of h, it holds

$$z_n := \eta_n x + (1-\eta_n)Tx \notin C$$

for any $n \in \mathbb{N}$. Since $\eta_n \to h(x)$ and

$$\|z_n - h(x)x - (1-h(x))Tx\| \le |\eta_n - h(x)|\|x - Tx\|,$$

it follows that $z_n \to h(x)x + (1-h(x))Tx \in C$, so that this last must belong to ∂C. □

Our main result is the following:

Theorem 1 *Let C be a convex, closed and nonempty subset of a Hilbert space H and let $T : C \to H$ be a mapping. Then, the algorithm*

$$\begin{cases} x_0 \in C, \\ \alpha_0 := \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} := \max\{\alpha_n, h(x_{n+1})\} \end{cases} \tag{4}$$

is well-defined.

If we further assume that

1. *C is strictly convex and*
2. *T is a non-expansive mapping, which satisfies the inward condition (2) and such that $Fix(T) \neq \emptyset$,*

then $\{x_n\}$ weakly converges to a point $p \in Fix(T)$. Moreover, if $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$ then the convergence is strong.

Proof To prove that the algorithm is well-defined, it is sufficient to note that $\alpha_n \in [h(x_n), 1]$ for any $n \in \mathbb{N}$; then, by recalling property $(P2)$ from Lemma 1, it immediately follows that

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \in C.$$

Assume now that T satisfies the inward condition. In this case, by the property $(P3)$ of the previous lemma, we obtain that the non-decreasing sequence$\{\alpha_n\}$ is contained in $(\frac{1}{2}, 1)$. Also, since T is non-expansive and with at least one fixed point, it follows by standard arguments that $\{x_n\}$ is Fejér-monotone with respect to $Fix(T)$ and, as a consequence, both $\{x_n\}$ and $\{Tx_n\}$ are bounded.

Firstly, assume that $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Then, since $\alpha_n \geq \frac{1}{2}$, we derive that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$ and from [16, Lemma 2] we obtain that

$$\|x_n - Tx_n\| \to 0.$$

This fact, together with the Fejér-monotonicity of $\{x_n\}$ proves that the sequence weakly converges in $Fix(T)$ (see [17, Proposition 2.1]).

Suppose that

$$\sum_{n=0}^{\infty}(1-\alpha_n) < \infty. \tag{5}$$

Since,

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|Tx_n - x_n\|,$$

and by the boundedness of $\{x_n\}$ and $\{Tx_n\}$, it is promptly obtained that

$$\sum_{n=0}^{\infty}\|x_{n+1} - x_n\| < \infty,$$

i.e. $\{x_n\}$ is a strongly Cauchy sequence and hence $x_n \to x^* \in C$.

Note that T satisfies the inward condition. Then by applying the properties $(P2)$ and $(P3)$ from Lemma 1, we obtain that $h(x^*) < 1$ and that for any $\mu \in (h(x^*), 1)$ it holds

$$\mu x^* + (1-\mu)Tx^* \in C. \tag{6}$$

On the other hand, we observe that since $\lim_{n\to\infty}\alpha_n = 1$ by (5) and since $\alpha_n = \max\{\alpha_{n-1}, h(x_n)\}$ holds, it follows that we can choose a sub-sequence $\{x_{n_k}\}$ with the property that $\{h(x_{n_k})\}$ is non-decreasing and $h(x_{n_k}) \to 1$. In particular, for any $\mu < 1$,

$$\mu x_{n_k} + (1-\mu)Tx_{n_k} \notin C \tag{7}$$

eventually holds.

Choose $\mu_1, \mu_2 \in (h(x^*), 1)$ with $\mu_1 \neq \mu_2$ and set $v_1 := \mu_1 x^* + (1-\mu_1)Tx^*$ and $v_2 := \mu_2 x^* + (1-\mu_2)Tx^*$. Then, whenever $\mu \in [\mu_1, \mu_2]$, by (6) we have that $v := \mu x^* + (1-\mu)Tx^* \in C$. Moreover,

$$\mu x_{n_k} + (1-\mu)Tx_{n_k} \to v,$$

since $x_n \to x^*$. This last, together with (7), implies that $v \in \partial C$ and $[v_1, v_2] \subset \partial C$, since μ is arbitrary.

By the strict convexity of C, we derive that

$$\mu_1 x^* + (1-\mu_1)Tx^* = \mu_2 x^* + (1-\mu_2)Tx^*$$

and $x^* = Tx^*$ must necessarily hold, i.e. $\{x_n\}$ strongly converges to a fixed point of T. $\square$

Remark 1 Following the same line of the proof, it can be easily seen that the same results hold true if the starting coefficient $\alpha_0 = \max\{\frac{1}{2}, h(x_0)\}$ is substituted by $\alpha_0 = \max\{b, h(x_0)\}$, where $b \in (0,1)$ is a fixed and arbitrary value. In the statement of Theorem 1, the value $b = \frac{1}{2}$ had been taken to ease the notation.

We also note that the value $h(x_n)$ can be replaced, in practice, by $h_n = 1 - \frac{1}{2^{j_n}}$, where $j_n := \min\{j \in \mathbb{N} : (1 - \frac{1}{2^j})x_n + \frac{1}{2^j}Tx_n \in C\}$.

Remark 2 As it follows from the proof, the condition $\sum_n (1-\alpha_n) < \infty$ provides a localization result for the fixed point x^* as a side result. Indeed, in this case, it holds that $x^* = v_1 = v_2$ belongs to the boundary ∂C of the set C.

Remark 3 In [18], for a closed and convex set C, the map

$$f(x) := \inf\{\lambda \in [0,1] : x \in \lambda C\}$$

had been introduced and used in conjunction with an iterative scheme to approximate a fixed point of minimum norm (see also [19]). Indeed, in the above mentioned paper, it is proved that the iterative scheme

$$\begin{cases} \lambda_n = \max\{f(x_n), \lambda_{n-1}\} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ x_{n+1} = \alpha_n \lambda_n x_n + (1-\alpha_n)y_n, \end{cases}$$

strongly converges under the assumptions that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_n \frac{\alpha_n}{(1-\lambda_n)} = 0$ and that $\sum_n (1-\lambda_n)\alpha_n = \infty$. We point out that the mentioned conditions appear to be difficult to be checked as they involve the geometry of the set C.

We illustrate the statement of our results with a brief example.

Example 1 Let $H = l^2(\mathbb{R})$ and let $C := B_1 \cap B_2$, where $B_1 := \{(t_i)_{i\in\mathbb{N}} : (t_1 - 49.995)^2 + \sum_{i=2}^{\infty} t_i^2 \leq (50.005)^2\}$ and $B_2 := \{(t_i)_{i\in\mathbb{N}} : \sum_{i=1}^{\infty} t_i^2 \leq 1\}$. then C is a non-empty, closed and strictly convex subset of H. Let $T : C \to H$ be the map defined by $T(t_1, t_2, \ldots, t_i, \ldots) := (-t_1, t_2, \ldots, t_i, \ldots)$, then T is a non-expansive inward map with $Fix(T) = \{(0, t_2, \ldots, t_i, \ldots) : \sum_{i=2}^{\infty} t_i^2 \leq 1\}$. If we use the algorithm

$$\begin{cases} x_0 = (t_i)_{i \in \mathbb{N}} \in C, \\ \alpha_0 := \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} := \alpha_n x_n + (1 - \alpha_n) T x_n, \\ \alpha_{n+1} := \max\{\alpha_n, h(x_{n+1})\}, \end{cases}$$

then, by the natural symmetry of the problem, we obtain the constant sequence

$$x_1 = \ldots = x_n = (0, t_2, \ldots, t_i, \ldots) \in Fix(T).$$

If we use the algorithm

$$\begin{cases} x_0 = (t_i)_{i \in \mathbb{N}} \in C, \\ \alpha_0 := \max\{0.01, h(x_0)\}, \\ x_{n+1} := \alpha_n x_n + (1 - \alpha_n) T x_n, \\ \alpha_{n+1} := \max\{\alpha_n, h(x_{n+1})\}, \end{cases}$$

then $\{x_n\}$ still converges in $Fix(T)$, but $\{x_n\} \cap Fix(T) = \emptyset$ whenever $t_i \neq 0$.

We conclude the paper by including few question that appears to be still open, to the best of our knowledge.

Question 1 It had been proved that the Krasnoselskii-Mann algorithm converges for general classes of mappings (see, e.g., [20] and [21]). By maintaining the same assumption on the set C and the inward condition of the involved map, it appears to be natural to ask for which classes of mappings the same result of Theorem 1 still holds.

Question 2 Under which assumptions the algorithm (4) can be adapted to produce a converging sequence to a common fixed point for a family of mappings? In other words, does the algorithm

$$\begin{cases} x_0 \in C, \\ \alpha_0 := \max\{\frac{1}{2}, h_n(x_0)\}, \\ x_{n+1} := \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ \alpha_{n+1} := \max\{\alpha_n, h_{n+1}(x_{n+1})\} \end{cases}$$

converge to a common fixed point of the family $\{T_n\}$, where

$$h_n(x) := \inf\{\lambda \geq 0 : \lambda x + (1 - \lambda) T_n x \in C\}$$

and under suitable hypotheses?

We refer to [22] and [23] for two examples regarding the classical Krasnoselskii-Mann algorithm.

Question 3 In the classical literature, it had been proved that the inward condition can be often dropped, in favor of weaker condition. For example, a mapping $T : C \to X$ is said to be weakly inward (or to satisfy the weakly inward condition) if

$$Tx \in \overline{I_C(x)} \qquad \forall x \in C.$$

Does Theorem 1 hold even for weakly inward mappings?

On the other hand, we observe that the strict convexity of the set C does appear to be unusual for results regarding the convergence of Krasnoselskii-Mann iterations. We do not know if our result can hold for a convex and closed set C, even at the price of strengthening the requirements on the map T.

Competing interests

The authors declare that they have no competing interests.

Authors' contribution

All authors contributed equally and significantly in writing the article. All authors read and approved the final manuscript.

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References

1. C. Chidume, Geometric properties of Banach spaces and nonlinear iterations, Vol. 1965, Springer, 2009.
2. W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (3) (1953) 506–510.
3. C. W. Groetsch, A note on segmenting Mann iterates, J. Math. Anal. Appl. 40 (2) (1972) 369–372.
4. T. L. Hicks, J. D. Kubicek, On the Mann iteration process in a Hilbert space, J. Math. Anal. Appl. 59 (3) (1977) 498–504.
5. M. Edelstein, R. C. OBrien, Nonexpansive mappings, asymptotic regularity and successive approximations, J. Lond. Math. Soc. 2 (3) (1978) 547–554.
6. B. P. Hillam, A generalization of Krasnoselski's theorem on the real line, Math. Mag. (1975) 167–168.
7. S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (2) (1979) 274–276.
8. H.-K. Xu, X.-M. Yin, Strong convergence theorems for nonexpansive nonself-mappings, Nonlinear Anal. 24 (2) (1995) 223–228.
9. H.-K. Xu, Approximating curves of nonexpansive nonself-mappings in Banach spaces, C. R. Acad. Sci. Paris Sér. I Math. 325 (2) (1997) 151–156.
10. G. Marino, G. Trombetta, On approximating fixed points for nonexpansive mappings, Indian J. Math 34 (1992) 91–98.
11. W. Takahashi, G.-E. Kim, Strong convergence of approximants to fixed points of nonexpansive nonself-mappings in Banach spaces, Nonlinear Anal. 32 (3) (1998) 447–454.
12. Y. Song, R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, J. Math. Anal. Appl. 321 (1) (2006) 316–326.
13. Y. S. Song, Y. J. Cho, Averaged iterates for non-expansive nonself mappings in Banach spaces, J. Comput. Anal. Appl. 11 (2009) 451–460.
14. H. Zhou, P. Wang, Viscosity approximation methods for nonexpansive nonself-mappings without boundary conditions, Fixed Point Theory Appl. 2014 (1) (2014) 61.
15. W. Kirk, B. Sims, Handbook of metric fixed point theory, Springer, 2001.
16. S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc. (1976) 65–71.
17. H. H. Bauschke, P. L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, Math. Oper. Res. 26 (2) (2001) 248–264.
18. S. He, W. Zhu, A Modified Mann Iteration by Boundary Point Method for Finding Minimum-Norm Fixed Point of Nonexpansive Mappings, Abstr. Appl. Anal. 2013 (2013).
19. S. He, C. Yang, Boundary point algorithms for minimum norm fixed points of nonexpansive mappings, Fixed Point Theory Appl. 2014 (1) (2014) 56.

20. J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (2) (1991) 407–413.
21. G. Marino, H.-K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (1) (2007) 336–346.
22. H. H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1) (1996) 150–159.
23. T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (1) (2005) 227–239.

REFEREE'S REPORT

TITLE: Kranosel'sliĭ-Mann method for non-self mappings

AUTHORS: V. Colao; G. Marino

COMMENTS: This is a very interesting paper about convergence of a kind of Kranosel'sliĭ-Mann iterations to a fixed point of a non-expansive mapping. The main originality of the paper is the use of non-self mapping. Surprisingly, in this more general setting, the authors are able to obtain a strong convergence result (usually we cannot expect to obtain the strong convergence for a Kranosel'sliĭ-Mann scheme) based upon the behavior of the mapping T and the geometry of its domain.

The paper is well written and can open new research lines on this topic. Thus, I recommend acceptance in Fixed Point Theory and Applications.

We include below a list with some misprints or observations:

Page 2, line -13.
It says: ... is that in the last the choice of...a priori,
It must say: ... is that the choice of the ...a priori in the latter,

Page 3, line 11
It says: For
It must say: for

Page 4, line 18
It says: This last,
It must say: This fact,

Page 5, line 5
Watch the parindent

Page 5, line -7
It says: thatlim
It must say: that lim

Page 6, line 2
It says: $\sum_{i=2}^{\infty} t_i$
It must say $\sum_{i=2}^{\infty} t_i^2$

Page 6, line 17
It says: Question 1 it
It should says: Question 1 It

Page 7, Reference 9
It says: -Sciences Series I-Mathematics
It must say: Sciences-Série I, Mathématique

Page 9, References.
Use standard abbreviations (as used in MathScinet)

Report on

Krasnoselskii-Mann method for non-self mappings

G. Marino and L. Muglia

The aim of this paper is to generalize the well-known Krasnoselskii - Mann algorithm defined on a closed and convex subset C of a Hilbert space H by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n. \tag{1}$$

where $x_o \in C$ is a fixed starting point, $\{\alpha_n\} \subset (0, 1)$ is a fixed sequence and $T : C \to C$ is a nonexpansive mapping which has fixed points.
The main result of this paper is Theorem 1, which is a generalization of the formula (1). More precisely mappings T considered in Theorem 2 are nonexpansive mappings defined on strictly convex, closed subsets of a Hilbert space. These mappings have fixed points but they are not self-mappings like in formula (1). Instead of that they satisfy so called invard condition (see p. 2 of the manuscript, Def. 1). For fixed $T : C \to H$ as above, where C is a nonempty strictly convex and closed subset of H, the authors define a function $h : C \to \mathbb{R}$ by

$$h(x) = \inf\{\lambda \geq 0 : \lambda x + (1 - \lambda)Tx \in C\}$$

Then having fixed $x_o \in C$ and $\alpha_o = \max\{1/2, h(x_o)\}$ they prove the weak convergence of the folowing scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

$$\alpha_{n+1} = \max\{\alpha_n, h_{x_{n+1}}\}.$$

to a certain fixed point of T Moreover under the assumption $\sum_{n=0}^{\infty}(1 - \alpha_n) < \infty$, the convergence of the above scheme is strong. The main difference between (1) and the method proposed in Theorem 1 is that the choice of coefficients $\{\alpha_n\}$ is not made a-priori.

In my opinion the results given in the paper under review are original, they present a good mathematical level and the proofs of them are correct. Hence I recommend the paper entitled *Krasnoselskii-Mann method for non-self mappings* to be published in the Fixed Point Theory and Applications.