Datalog with Negation

Wolfgang Faber

University of Calabria, Italy

2007

Nonmonotonic Queries

- Some simple queries cannot be written in positive Datalog.
- Example: $(\pi_1 \ R) - S$
- This query is nonmonotone!
- Adding tuples to S may retract result tuples.
- Positive Datalog can express only monotone queries.

Nonmonotonic Queries

- In Relational Calculus $(\pi_1\ R) - S$ is written using negation.
- Introduce negation also for Datalog!
- Problem: Negation through recursion?

Outline

Closed World Assumption

- Atoms for which it is not necessary to be true should be considered as false.
- Only those items which are known should be true.
- Example: Timetable
- Reason for Minimal Model semantic!

Closed World Assumption

Definition

For a positive program $\mathcal{P}$, $CWA(\mathcal{P}) = \{A \mid \mathcal{P} \not\models A\}$.
Equivalently: $CWA(\mathcal{P}) = \mathbf{HB}(\mathcal{P}) - MM(\mathcal{P})$

Is this as simple if we allow rules with negative body literals?

Outline

Normal Programs – Syntax

Definition

A normal rule is

$$h \leftarrow b_1, \ldots, b_m, \mathtt{not}\ b_{m+1}, \ldots, \mathtt{not}\ b_n.$$
$$1 \leq m \leq n$$

Let
$B^+(r) = \{b_1, \ldots, b_m\}$
$B^-(r) = \{b_{m+1}, \ldots, b_m\}$
$\mathtt{not}.a = \mathtt{not}\ a, \mathtt{not}.\mathtt{not}\ a = a$
$\mathtt{not}.L = \{\mathtt{not}.l \mid l \in L\}$
$B(r) = B^+(r) \cup \mathtt{not}.B^-(r)$
$H(r), V(r), C(r)$ as before

Unsafe Queries

Recall: Using Negation it is easy to violate domain independence!

Example

$$positive(X) \leftarrow \texttt{not}\ zero(X).$$

Definition (Safety)

Each variable in a rule must occur in a positive body atom.

Example

$$answer(X) \leftarrow mynumber(X), \texttt{not}\ zero(X).$$

Unsafe Queries

Recall: Using Negation it is easy to violate domain independence!

Example

$$positive(X) \leftarrow \texttt{not}\ zero(X).$$

Definition (Safety)

Each variable in a rule must occur in a positive body atom.

Example

$$answer(X) \leftarrow mynumber(X), \texttt{not}\ zero(X).$$

Outline

Normal Programs – Semantics

- Most concepts do not change.
- Satisfaction of a rule r with respect to M:
  If $B^+(r) \subseteq M$ and $M \cap B^-(r) = \emptyset$, then $H(r) \in M$
- Question: Minimal Model semantics suitable?

Normal Programs

In general there is no unique minimal model.

Example

$$a \leftarrow \texttt{not}\, b.$$

There are two models $M_1 = \{a\}$ und $M_2 = \{b\}$.
M_2 is not very intuitive.

Normal Programs

In general there is no unique minimal model.

Example

$$a \leftarrow \texttt{not}\, b.$$

There are two models $M_1 = \{a\}$ und $M_2 = \{b\}$.
M_2 is not very intuitive.

Normal Programs

In general there is no unique minimal model.

Example

$$a \leftarrow \texttt{not}\, b.$$

There are two models $M_1 = \{a\}$ und $M_2 = \{b\}$.
M_2 is not very intuitive.

Normal Programs

Semantics of “negative recursion”?

$$
\begin{aligned}
&person(nicola).\\
&male(X) \leftarrow person(X), \texttt{not}\ female(X).\\
&female(X) \leftarrow person(X), \texttt{not}\ male(X).
\end{aligned}
$$

$\{person(nicola), male(nicola)\}$ and $\{person(nicola), female(nicola)\}$ are minimal models

Both are equally intuitive.

Normal Programs

Semantics of “negative recursion”?

$$\begin{array}{l} \mathit{person}(\mathit{nicola}). \\ \mathit{male}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{female}(X). \\ \mathit{female}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{male}(X). \end{array}$$

$\{\mathit{person}(\mathit{nicola}), \mathit{male}(\mathit{nicola})\}$ and
$\{\mathit{person}(\mathit{nicola}), \mathit{female}(\mathit{nicola})\}$ are minimal models
Both are equally intuitive.

Normal Programs

Semantics of “negative recursion”?

$$\begin{array}{l} \mathit{person}(\mathit{nicola}). \\ \mathit{male}(X) \leftarrow \mathit{person}(X), \texttt{not}\ \mathit{female}(X). \\ \mathit{female}(X) \leftarrow \mathit{person}(X), \texttt{not}\ \mathit{male}(X). \end{array}$$

$\{\mathit{person}(\mathit{nicola}), \mathit{male}(\mathit{nicola})\}$ and
$\{\mathit{person}(\mathit{nicola}), \mathit{female}(\mathit{nicola})\}$ are minimal models
Both are equally intuitive.

Possibilities

1. Pragmatic: Do not allow “recursion through negation”.
2. Three-valued: Stay with a unique model, which may leave some atoms undefined.
3. Two-valued: Abandon model uniqueness, stay with standard models.

Outline

Dependency Graph

Definition

For a negative Datalog program $\mathcal{P}$, we define a directed graph (V, E), where V are the predicate symbols of $\mathcal{P}$, and $(p, q) \in E$ if p is in the head and q is in the body of some rule. If q is in the negative body, we mark the arc.

Examples

Example

$$a \leftarrow b.$$
$$c \leftarrow \texttt{not}\ b.$$
$$b \leftarrow a$$

Example

$$a \leftarrow b, c.$$
$$c \leftarrow \texttt{not}\ b.$$
$$b \leftarrow a$$

Examples

Example

$$a \leftarrow b.$$
$$c \leftarrow \mathtt{not}\ b.$$
$$b \leftarrow a$$

Example

$$a \leftarrow b, c.$$
$$c \leftarrow \mathtt{not}\ b.$$
$$b \leftarrow a$$

Outline

Stratification

Main idea: Partition the program along negation.

Definition

A stratification is a function λ, which maps predicate symbols to integers such that for each rule with p being the head predicate the following conditions hold:

1. For each predicate q in the positive body, $\lambda(p) \geq \lambda(q)$.
2. For each predicate r in the negative body, $\lambda(p) > \lambda(r)$.

Stratification

- λ induces a partition $\langle P_0, \ldots, P_n \rangle$ of $\mathcal{P}$ (assuming that λ maps to integers between 0 and n):

$$P_0 = \{r \mid \lambda(H(r)) = 0$$
$$\ldots$$
$$P_n = \{r \mid \lambda(H(r)) = n$$

- λ defines a partial ordering between partitions.
- We can evaluate the program along this ordering.

Examples

Example

$$a \leftarrow b.$$
$$c \leftarrow \texttt{not}\ b.$$
$$b \leftarrow a$$

Stratifiable: $\lambda(a) = 0, \lambda(b) = 0, \lambda(c) = 1$

Example

$$a \leftarrow b, c.$$
$$c \leftarrow \texttt{not}\ b.$$
$$b \leftarrow a$$

Not stratifiable: $\lambda(c) > \lambda(b) \geq \lambda(a) \geq \lambda(c)$

Examples

Example

$$a \leftarrow b.$$
$$c \leftarrow \texttt{not}\ b.$$
$$b \leftarrow a$$

Stratifiable: $\lambda(a) = 0, \lambda(b) = 0, \lambda(c) = 1$

Example

$$a \leftarrow b, c.$$
$$c \leftarrow \texttt{not}\ b.$$
$$b \leftarrow a$$

Not stratifiable: $\lambda(c) > \lambda(b) \geq \lambda(a) \geq \lambda(c)$

Examples

Example

$$a \leftarrow b.$$
$$c \leftarrow \texttt{not } b.$$
$$b \leftarrow a$$

Stratifiable: $\lambda(a) = 0, \lambda(b) = 0, \lambda(c) = 1$

Example

$$a \leftarrow b, c.$$
$$c \leftarrow \texttt{not } b.$$
$$b \leftarrow a$$

Not stratifiable: $\lambda(c) > \lambda(b) \geq \lambda(a) \geq \lambda(c)$

Examples

Example

$$a \leftarrow b.$$
$$c \leftarrow \mathtt{not}\ b.$$
$$b \leftarrow a$$

Stratifiable: $\lambda(a) = 0, \lambda(b) = 0, \lambda(c) = 1$

Example

$$a \leftarrow b, c.$$
$$c \leftarrow \mathtt{not}\ b.$$
$$b \leftarrow a$$

Not stratifiable: $\lambda(c) > \lambda(b) \geq \lambda(a) \geq \lambda(c)$

Stratification

Theorem

A program is stratifiable if and only if its dependency graph contains no cycle with a marked ("negative") edge.

Outline

Perfect Models

- Stratification specifies an order for evaluation.
- First fully compute the relations in the lowest stratum.
- Then move one stratum up and evaluate the relations there.
- Negation is evaluated only over fully computed relations.
- Can be treated like negation over EDB predicates.

Perfect Models und $\mathbf{T}_{\mathcal{P}}$

Modify operator $\mathbf{T}_{\mathcal{P}}$, as $\mathcal{P}$ may contain negation.

Definition

$$\mathbf{T}_{\mathcal{P}}(I) = \{h \,|\, r \in Ground(\mathcal{P}), B^{+}(r) \subseteq I, h \in H(r), \\ \texttt{not}.B^{-}(r) \cap I = \emptyset\} \cup I$$

Perfect Models und $\mathbf{T}_{\mathcal{P}}$

Definition

Let $\langle P_0, \ldots, P_n \rangle$ be the partitions of a stratifiable program $\mathcal{P}$, induced by a stratification λ.
The sequence $M_0 = \mathbf{T}^{\infty}_{P_0}(\emptyset)$, $M_1 = \mathbf{T}^{\infty}_{P_1}(M_0)$, $\ldots$, $M_n = \mathbf{T}^{\infty}_{P_n}(M_{n-1})$ defines the Perfect Model M_n of $\mathcal{P}$.

Example – stratifiable

Easy case: Negation only on EDB predicates

Example

$$color(yellow, k1).\ color(yellow, k2).\ color(blue, k3).$$
$$color(green, k4).\ color(red, k5).$$

$$block(K) \leftarrow color(F, K). \quad block(K) \leftarrow form(F, K).$$
$$diffcolor(K1, K2) \leftarrow$$
$$color(F, K1), block(K2), \mathrm{not}\ color(F, K2).$$

Example – stratifiable

Example

$form(box, k1).\ form(cone, k2).\ form(disc, k3).$
$form(box, k4).\ form(pyramid, k5).$

$block(K) \leftarrow color(F, K).\quad block(K) \leftarrow form(F, K).$
$pointy\_top(K) \leftarrow block(K), form(cone, K).$
$pointy\_top(K) \leftarrow block(K), form(pyramid, K).$
$fits\_on(K1, K2) \leftarrow block(K1), block(K2), \texttt{not}\ pointy\_top(K2).$

Example – stratifiable

Example

$form(box, k1).\ form(cone, k2).\ form(disc, k3).$
$form(box, k4).\ form(pyramid, k5).$

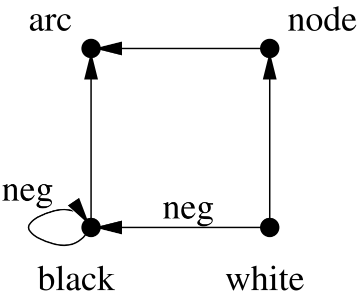
$block(K) \leftarrow color(F, K). \quad block(K) \leftarrow form(F, K).$
$flat\_top(K) \leftarrow block(K), form(box, K).$
$flat\_top(K) \leftarrow block(K), form(disc, K).$
$pointy\_top(K) \leftarrow block(K), \texttt{not}\ flat\_top(K).$
$fits\_on(K1, K2) \leftarrow block(K1), block(K2), \texttt{not}\ pointy\_top(K2).$

Example – unstratified

$$arc(a,b).\ arc(b,c).\ arc(b,d).$$
$$node(N) \leftarrow arc(N,Y).\ node(N) \leftarrow arc(X,N).$$
$$black(Y) \leftarrow arc(X,Y), \mathtt{not}\ black(X).$$
$$white(X) \leftarrow node(X), \mathtt{not}\ black(X).$$

Example – unstratified

Dependency Graph:



Perfect Models

- Note: Perfect Models are defined only on stratifiable programs.

Theorem

For any stratifiable program, there exists a unique Perfect Model.

Unstratifiable Programs

Example

$$person(nicola).$$
$$alive(X) \leftarrow person(X).$$
$$male(X) \leftarrow person(X), \texttt{not}\ female(X).$$
$$female(X) \leftarrow person(X), \texttt{not}\ male(X).$$

Perfect Models are not defined.
But we would like to conclude at least $alive(nicola)$.

Unstratifiable Programs

Example

$$\mathit{person}(\mathit{nicola}).$$
$$\mathit{alive}(X) \leftarrow \mathit{person}(X).$$
$$\mathit{male}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{female}(X).$$
$$\mathit{female}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{male}(X).$$

Perfect Models are not defined.

But we would like to conclude at least *alive*(*nicola*).

Unstratifiable Programs

Example

$$\begin{array}{l} \mathit{person}(\mathit{nicola}). \\ \mathit{alive}(X) \leftarrow \mathit{person}(X). \\ \mathit{male}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{female}(X). \\ \mathit{female}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{male}(X). \end{array}$$

Perfect Models are not defined.
But we would like to conclude at least $\mathit{alive}(\mathit{nicola})$.

Recursive Negation

Example

$$
\begin{array}{l}
\mathit{person}(\mathit{nicola}).\\
\mathit{alive}(X) \leftarrow \mathit{person}(X).\\
\mathit{male}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{female}(X).\\
\mathit{female}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{male}(X).
\end{array}
$$

Recursive Negation

Example

Using generalized $\mathbf{T}_{\mathcal{P}}$:

$$\begin{array}{l}
\mathbf{T}_{\mathcal{P}}(\emptyset) = \{\mathit{person}(\mathit{nicola})\} \\
\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\emptyset)) = \{\mathit{person}(\mathit{nicola}), \mathit{alive}(\mathit{nicola}), \mathit{male}(\mathit{nicola}), \mathit{female}(\mathit{nicola})\} \\
\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\emptyset))) = \{\mathit{person}(\mathit{nicola}), \mathit{alive}(\mathit{nicola})\} \\
\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\emptyset)))) = \mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\emptyset)) \\
\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\mathbf{T}_{\mathcal{P}}(\emptyset))))) = \mathbf{T}_{\mathcal{P}}(\emptyset) \\
\ldots
\end{array}$$

Recursive Negation

Example

But there are two fixpoints:

$$\mathbf{T}_{\mathcal{P}}(\{person(nicola), alive(nicola), male(nicola)\}) = \{person(nicola), alive(nicola), male(nicola)\}$$

$$\mathbf{T}_{\mathcal{P}}(\{person(nicola), alive(nicola), female(nicola)\}) = \{person(nicola), alive(nicola), female(nicola)\}$$

Recursive Negation

Two ways of resolving this:

1. Be cautious and do not say anything about *male*(*nicola*) and *female*(*nicola*).
2. Consider two scenarios: One in which *male*(*nicola*) is true, another in which *female*(*nicola*) is true.

Problems to resolve:

1. needs another truth value undefined.
2. allows more than one model.

Recursive Negation

Two ways of resolving this:

1. Be cautious and do not say anything about *male*(*nicola*) and *female*(*nicola*).
2. Consider two scenarios: One in which *male*(*nicola*) is true, another in which *female*(*nicola*) is true.

Problems to resolve:

1. needs another truth value undefined.
2. allows more than one model.

Outline

Three-valued Interpretations

Definition

A three-valued (or partial) interpretation I is a set of ground `not` literals, such that for any ground atom a not both $a \in I$ and $\texttt{not}\, a \in I$.

Example

$I = \{\texttt{not}\ a, c\}$

- a is false in I
- b is undefined in I
- c is true in I

Three-valued Interpretations

Definition

A three-valued (or partial) interpretation I is a set of ground $\mathtt{not}$ literals, such that for any ground atom a not both $a \in I$ and $\mathtt{not}\, a \in I$.

Example

$I = \{\mathtt{not}\ a, c\}$

- a is false in I
- b is undefined in I
- c is true in I

Three-valued Interpretations

Definition

A three-valued (or partial) interpretation I is a set of ground $\texttt{not}$ literals, such that for any ground atom a not both $a \in I$ and $\texttt{not}\,a \in I$.

Example

$I = \{\texttt{not}\ a, c\}$

- a is false in I
- b is undefined in I
- c is true in I

Unfounded Sets

Goal: Derive as much negative information as possible.

Example

$$a \leftarrow \texttt{not}\ b.$$

b does not occur in any head, thus can never become true und should be false. a should therefore be true.

Unfounded Sets

Goal: Derive as much negative information as possible.

Example

$$a \leftarrow b.$$
$$c \leftarrow \texttt{not}\ a.$$

Given the interpretation $\{\texttt{not}\ b\}$, a can never become true and should be false. c should be true in this case.

Unfounded Sets

Goal: Derive as much negative information as possible.

Example

$$a \leftarrow b.$$
$$b \leftarrow a.$$
$$c \leftarrow \texttt{not}\ a.$$

a and *b* occur in some heads, but all bodies of these rules require one of *a* or *b* to become true. Therefore *a* and *b* can become true only via themselves and should be false, hence *c* should be true.

Unfounded Sets – Definition

Definition

A set $U \subseteq \mathbf{HB}(\mathcal{P})$ is unfounded with respect to a partial interpretation I if the following holds:
For each $a \in U$ and each rule $r \in Ground(\mathcal{P})$ with $H(r) = \{a\}$ at least one of the the following conditions holds:

1. $\exists \ell \in B(r) : \mathtt{not}.\ell \in I$
2. $B^+(r) \cap U \neq \emptyset$

Unfounded Sets – Example

Example

$$a \leftarrow \mathtt{not}\ b.$$

For $I = \emptyset$, $\{b\}$ is an unfounded set.

Unfounded Sets – Example

Example

$$a \leftarrow b.$$
$$c \leftarrow \mathtt{not}\ a.$$

For $I = \{\mathtt{not}\ b\}$, $\{a\}$ is an unfounded set.

Unfounded Sets – Example

Example

$$a \leftarrow b.$$
$$b \leftarrow a.$$
$$c \leftarrow \mathtt{not}\ a.$$

For $I = \emptyset$, $\{a, b\}$ is an unfounded set, because condition 2 holds for $a \leftarrow b.$ and $b \leftarrow a.$.

Outline

Unfounded Operator

Theorem

For any program $\mathcal{P}$ and partial interpretation I, the greatest unfounded set $GUS_{\mathcal{P}}(I)$ (which is a superset of all unfounded sets) exists and is unique.

Idea: Use $GUS_{\mathcal{P}}(I)$ to derive negative information.

Definition

Operator $\mathbf{U}_{\mathcal{P}}(I) = \{\texttt{not}.a \mid a \in GUS_{\mathcal{P}}(I)\}$

Unfounded Operator

Theorem

For any program $\mathcal{P}$ and partial interpretation I, the greatest unfounded set $GUS_{\mathcal{P}}(I)$ (which is a superset of all unfounded sets) exists and is unique.

Idea: Use $GUS_{\mathcal{P}}(I)$ to derive negative information.

Definition

Operator $\mathbf{U}_{\mathcal{P}}(I) = \{\mathtt{not}.a \mid a \in GUS_{\mathcal{P}}(I)\}$

Well-Founded Operator

First generalize $\mathbf{T}_{\mathcal{P}}(I)$ for partial interpretations:

Definition

$\mathbf{T}_{\mathcal{P}}(I) := \{h \mid r \in Ground(\mathcal{P}), B(r) \subseteq I, h \in H(r)\}$

Define the well-founded operator $\mathbf{W}_{\mathcal{P}}(I)$ as a combination of $\mathbf{T}_{\mathcal{P}}(I)$ and $\mathbf{U}_{\mathcal{P}}(I)$.

Definition

$\mathbf{W}_{\mathcal{P}}(I) = \mathbf{T}_{\mathcal{P}}(I) \cup \mathbf{U}_{\mathcal{P}}(I)$

Well-Founded Operator

First generalize $\mathbf{T}_{\mathcal{P}}(I)$ for partial interpretations:

Definition

$$\mathbf{T}_{\mathcal{P}}(I) := \{h \mid r \in Ground(\mathcal{P}), B(r) \subseteq I, h \in H(r)\}$$

Define the well-founded operator $\mathbf{W}_{\mathcal{P}}(I)$ as a combination of $\mathbf{T}_{\mathcal{P}}(I)$ and $\mathbf{U}_{\mathcal{P}}(I)$.

Definition

$$\mathbf{W}_{\mathcal{P}}(I) = \mathbf{T}_{\mathcal{P}}(I) \cup \mathbf{U}_{\mathcal{P}}(I)$$

Well-Founded Model

Allen Van Gelder

Kenneth Ross

John Schlipf

Well-Founded Model

Theorem

$\mathbf{W}_{\mathcal{P}}$ *is monotone and allows for a least fixpoint.*

Definition

The least fixpoint $\mathbf{W}_{\mathcal{P}}^{\infty}(\emptyset)$ is the Well-Founded Model of a normal program $\mathcal{P}$.

Well-Founded Model

Theorem

$\mathbf{W}_{\mathcal{P}}$ *is monotone and allows for a least fixpoint.*

Definition

The least fixpoint $\mathbf{W}_{\mathcal{P}}^{\infty}(\emptyset)$ is the Well-Founded Model of a normal program $\mathcal{P}$.

Well-Founded Model – Properties

Theorem

Each normal program has a unique Well-Founded Model.

Well-Founded Model – Properties

Definition

A partial interpretation I is total if $I \cup \texttt{not}.I = \mathbf{HB}(\mathcal{P})$ (each ground atom is true or false).

Theorem

The Well-Founded Model for positive programs is total and corresponds to its Minimal Model.

Theorem

The Well-Founded Model for stratifiable programs is total and corresponds to its Perfect Model.

Well-Founded Model – Properties

Definition

A partial interpretation I is total if $I \cup \texttt{not}.I = \mathbf{HB}(\mathcal{P})$ (each ground atom is true or false).

Theorem

The Well-Founded Model for positive programs is total and corresponds to its Minimal Model.

Theorem

The Well-Founded Model for stratifiable programs is total and corresponds to its Perfect Model.

Well-Founded Model – Example

Example

$$\begin{array}{l} \mathit{person}(\mathit{nicola}). \\ \mathit{alive}(X) \leftarrow \mathit{person}(X). \\ \mathit{male}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{female}(X). \\ \mathit{female}(X) \leftarrow \mathit{person}(X), \mathtt{not}\ \mathit{male}(X). \end{array}$$

The Well-Founded Model is $\{\mathit{person}(\mathit{nicola}), \mathit{alive}(\mathit{nicola})\}$ and it is not total.

Stable Models

- No longer a unique model.
- Use total models.
- Stability criterion instead of fixpoint semantics.

Stable Models

Michael Gelfond

Vladimir Lifschitz

Stable Models

Nicole Bidoit

Christine Froidevaux

Outline

Gelfond-Lifschitz Reduct

Definition

The Gelfond-Lifschitz reduct of a program $\mathcal{P}^I$ is defined as follows, starting from $Ground(\mathcal{P})$:

1. Delete rules r, for which $B^-(r) \cap I \neq \emptyset$.
2. Delete the negative bodies of the remaining rules.

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\quad female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{male(g).\ female(g).\}$
$I_2 = \{male(g)\}, \mathcal{P}^{I_2} = \{male(g).\}$
$I_3 = \{female(g)\}, \mathcal{P}^{I_3} = \{female(g).\}$
$I_4 = \{male(g), female(g)\}, \mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\quad female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$
$I_4 = \{male(g), female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ \mathit{male}(g) \leftarrow \texttt{not}\ \mathit{female}(g).\quad \mathit{female}(g) \leftarrow \texttt{not}\ \mathit{male}(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{\mathit{male}(g).\ \mathit{female}(g).\}$

$I_2 = \{\mathit{male}(g)\}, \mathcal{P}^{I_2} = \{\mathit{male}(g).\}$

$I_3 = \{\mathit{female}(g)\}, \mathcal{P}^{I_3} = \{\mathit{female}(g).\}$

$I_4 = \{\mathit{male}(g), \mathit{female}(g)\}, \mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ \mathit{male}(g) \leftarrow \texttt{not}\ \mathit{female}(g).\quad \mathit{female}(g) \leftarrow \texttt{not}\ \mathit{male}(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{\mathit{male}(g).\ \mathit{female}(g).\}$
$I_2 = \{\mathit{male}(g)\}, \mathcal{P}^{I_2} = \{\mathit{male}(g).\}$
$I_3 = \{\mathit{female}(g)\}, \mathcal{P}^{I_3} = \{\mathit{female}(g).\}$
$I_4 = \{\mathit{male}(g), \mathit{female}(g)\}, \mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\quad female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{male(g).\ female(g).\}$
$I_2 = \{male(g)\}, \mathcal{P}^{I_2} = \{male(g).\}$
$I_3 = \{female(g)\}, \mathcal{P}^{I_3} = \{female(g).\}$
$I_4 = \{male(g), female(g)\}, \mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\\ female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$
$I_4 = \{male(g), female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{\ \mathit{male}(g) \leftarrow \mathtt{not}\ \mathit{female}(g).\quad \mathit{female}(g) \leftarrow \mathtt{not}\ \mathit{male}(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\mathit{male}(g).\ \mathit{female}(g).\}$

$I_2 = \{\mathit{male}(g)\}$, $\mathcal{P}^{I_2} = \{\mathit{male}(g).\}$

$I_3 = \{\mathit{female}(g)\}$, $\mathcal{P}^{I_3} = \{\mathit{female}(g).\}$

$I_4 = \{\mathit{male}(g), \mathit{female}(g)\}$, $\mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{ \ male(g) \leftarrow \texttt{not} \ female(g). \\ female(g) \leftarrow \texttt{not} \ male(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{male(g).\ female(g).\}$
$I_2 = \{male(g)\}, \mathcal{P}^{I_2} = \{male(g).\}$
$I_3 = \{female(g)\}, \mathcal{P}^{I_3} = \{female(g).\}$
$I_4 = \{male(g), female(g)\}, \mathcal{P}^{I_4} = \emptyset$

Gelfond-Lifschitz Reduct

Example

$$\mathcal{P} = \{ \ male(g) \leftarrow \texttt{not} \ female(g). \\ female(g) \leftarrow \texttt{not} \ male(g). \}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$
$I_4 = \{male(g), female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$

Stable Models

Fact

Gelfond-Lifschitz reducts are always positive, and have a unique Minimal Model.

Definition

A total interpretation M is a Stable Model of $\mathcal{P}$, if $M = MM(\mathcal{P}^M)$.

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \mathtt{not}\ female(g).$$
$$female(g) \leftarrow \mathtt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\quad female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).$$
$$female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{male(g)\}, \mathcal{P}^{I_2} = \{male(g).\}, MM(\mathcal{P}^{I_2}) = I_2$

I_2 is a stable model.

$I_3 = \{female(g)\}, \mathcal{P}^{I_3} = \{female(g).\}, MM(\mathcal{P}^{I_3}) = I_3$

I_3 is a stable model.

$I_4 = \{male(g).\ female(g)\}, \mathcal{P}^{I_4} = \emptyset, MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\quad female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$

I_2 is a stable model.

$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$

I_3 is a stable model.

$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \mathtt{not}\ female(g).$$
$$female(g) \leftarrow \mathtt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\quad female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).$$
$$female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$

I_2 is a stable model.

$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$

I_3 is a stable model.

$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).\\ female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \mathtt{not}\ female(g).\quad female(g) \leftarrow \mathtt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).$$
$$female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}$, $\mathcal{P}^{I_2} = \{male(g).\}$, $MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}$, $\mathcal{P}^{I_3} = \{female(g).\}$, $MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}$, $\mathcal{P}^{I_4} = \emptyset$, $MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{\ male(g) \leftarrow \texttt{not}\ female(g).$$
$$female(g) \leftarrow \texttt{not}\ male(g).\}$$

$I_1 = \emptyset, \mathcal{P}^{I_1} = \{male(g).\ female(g).\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{male(g)\}, \mathcal{P}^{I_2} = \{male(g).\}, MM(\mathcal{P}^{I_2}) = I_2$
I_2 is a stable model.
$I_3 = \{female(g)\}, \mathcal{P}^{I_3} = \{female(g).\}, MM(\mathcal{P}^{I_3}) = I_3$
I_3 is a stable model.
$I_4 = \{male(g).\ female(g)\}, \mathcal{P}^{I_4} = \emptyset, MM(\mathcal{P}^{I_4}) \neq I_4$

Stable Models – Example

Example

$$\mathcal{P} = \{ \mathit{weird} \leftarrow \mathtt{not}\ \mathit{weird}. \}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\mathit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{\mathit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{ \, \mathit{weird} \leftarrow \texttt{not} \; \mathit{weird}. \}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\mathit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{\mathit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{ \ \mathit{weird} \leftarrow \mathtt{not} \ \mathit{weird}. \}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\mathit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{\mathit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{\ \textit{weird} \leftarrow \texttt{not}\ \textit{weird}.\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\textit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{\textit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{\ weird \leftarrow \mathtt{not}\ weird.\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{weird.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{weird\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{\ \mathit{weird} \leftarrow \mathtt{not}\ \mathit{weird}.\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\mathit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{\mathit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{\ \mathit{weird} \leftarrow \texttt{not}\ \mathit{weird}.\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\mathit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$

$I_2 = \{\mathit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$

There is no stable model!

Stable Models – Example

Example

$$\mathcal{P} = \{\ \textit{weird} \leftarrow \texttt{not}\ \textit{weird}.\}$$

$I_1 = \emptyset$, $\mathcal{P}^{I_1} = \{\textit{weird}.\}$, $MM(\mathcal{P}^{I_1}) \neq I_1$
$I_2 = \{\textit{weird}\}$, $\mathcal{P}^{I_2} = \emptyset$, $MM(\mathcal{P}^{I_2}) \neq I_2$
There is no stable model!

Stable Models

Theorem

For positive programs there is exactly one Stable Model, which is equal to the Minimal Model.

Theorem

For stratifiable programs there is exactly one Stable Model, which is equal to the Perfect Model.

Stable Models

Theorem

For positive programs there is exactly one Stable Model, which is equal to the Minimal Model.

Theorem

For stratifiable programs there is exactly one Stable Model, which is equal to the Perfect Model.

Stable Models

Theorem

If the Well-Founded Model of a program is total, then the program has a corresponding unique Stable Model.

Theorem

The positive part of the Well-Founded Model of a program is contained in each Stable Model of the program.

Stable Models

Theorem

If the Well-Founded Model of a program is total, then the program has a corresponding unique Stable Model.

Theorem

The positive part of the Well-Founded Model of a program is contained in each Stable Model of the program.

Stable Models – Consequences

Definition (Brave/Credulous Reasoning)

$\mathcal{P} \models_b l$, if l is true in some Stable Model of $\mathcal{P}$.

Definition (Cautious/Skeptical Reasoning)

$\mathcal{P} \models_c l$, if l is true in all Stable Models of $\mathcal{P}$.

Note: If $\mathcal{P}$ admits no Stable Model, then all literals are cautious/skeptical consequences!

Stable Models – Consequences

Definition (Brave/Credulous Reasoning)

$\mathcal{P} \models_b l$, if l is true in some Stable Model of $\mathcal{P}$.

Definition (Cautious/Skeptical Reasoning)

$\mathcal{P} \models_c l$, if l is true in all Stable Models of $\mathcal{P}$.

Note: If $\mathcal{P}$ admits no Stable Model, then all literals are cautious/skeptical consequences!

Stable Models – Example

Example (Two-Colorability)

Given a graph, can each vertex be assigned one of two colors, such that adjacent vertices do not have the same color?

$$
\begin{array}{l}
\mathit{vertex}(V) \leftarrow \mathit{arc}(V, Y).\ \mathit{vertex}(V) \leftarrow \mathit{arc}(X, V).\\
\mathit{color}(V, \mathit{white}) \leftarrow \mathit{vertex}(V), \mathtt{not}\ \mathit{color}(V, \mathit{black}).\\
\mathit{color}(V, \mathit{black}) \leftarrow \mathit{vertex}(V), \mathtt{not}\ \mathit{color}(V, \mathit{white}).\\
\mathit{bad} \leftarrow \mathit{color}(V1, F), \mathit{color}(V2, F),\\
\qquad \mathit{arc}(V1, V2), \mathtt{not}\ \mathit{bad}.
\end{array}
$$