Logica del Primo Ordine 1 First-Order Logic 1

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Wolfgang Faber First-Order Logic 1

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Motivation Syntax

1 **Motivation**

- Why "more than" Propositional Logic?
- Intuition



- Terms
- Formulas



- Structures
- Valuation

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Motivation Syntax

Why "more than" Propositional Logic? Intuition

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Outline



• Why "more than" Propositional Logic?

- Intuition
- 2 Syntax
 - Terms
 - Formulas
- 3 Semantics
 - Structures
 - Valuation

Why "more than" Propositional Logic? Intuition

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Propositional Logic \Rightarrow Done?

We want to represent:

- Socrates is a human.
- Humans are mortal.

From this, we want to draw the conclusion:

• Socrates is mortal.

Why "more than" Propositional Logic? Intuition

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Propositional Logic \Rightarrow Done?

In propositional logic:

- variable SH (Socrates is a human.)
- variable HM (Humans are mortal.)
- variable *SM* (Socrates is mortal.)
- formula $F: SH \land$ ("Socrates is a human" and ...)
- SH → SM ("Socrates is a human" implies "Socrates is mortal.")

- $F \models SM$ ("Socrates is mortal.")
- ... and where is HM??
- This is not what we wanted to express!

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Propositional Logic \Rightarrow Done?

- Marco is a human.
- Humans are mortal.
- Marco is mortal.
- We talk about "objects," not about propositions!
- But: There are no objects in propositional logic.

Why "more than" Propositional Logic? Intuition

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Towards First-Order Logic

• "Start" from Propositional Logic.

- "Atoms" are no longer indivisible.
- Compose "atoms" from:
 - terms (representing objects) and
 - predicates (statements about terms).
- Socrates \Rightarrow term
- Mortal ⇒ predicate

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Functions

Socrates' father"

- ... represents exactly one "object,"
- but we refer to it from another "object."
- Function symbols map some "objects" to another "object".
- "Objects" are constant function symbols!

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Variable

"Humans are mortal"

- ... if some "object" is human, it is mortal.
- ... We need a variable ranging over "objects."
- Note: Such variables are completely different from propositional variables!
- Think about them as object variables.

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Quantifiers

"Humans are mortal"

- ... actually wants to express "all humans are mortal,"
- ... and not "some humans are mortal,"
- ... and certainly not "no humans are mortal."
- Quantifiers express
 - "for all objects" or
 - 2 "some object exists."

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Terms Formulas

Outline



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Terms Formulas

Function Symbols, Constants

• Countable set *F* of function symbols.

- An arity (nonnegative integer) is associated with each function symbol.
- Function symbols with arity 0 are constants ("objects").
- Examples:
 - socrates (arity 0)
 - father (arity 1)
 - *son* (arity 2)
 - supercalifragilistichespiralidoso287 (arity 6)

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Terms Formulas

Variables

- Countable set V of (object) variables.
- V and F are disjoint!
- Examples:
 - Human
 - Xiknve
 - *A*, *B*, *C*, *D*

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Terms

Build terms from function symbols and variables, respecting arities.

Again: Inductive definition.

- If v is a variable symbol, v is a term.
- If *f* is a function symbol of arity 0, *f* is a term.
- If *f* is a function symbol of arity *n* > 0, and *t*₁,..., *t_n* are terms, then *f*(*t*₁,..., *t_n*) is a term.

Ground term: Term not containing any variable.

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Terms Formulas

Terms – Examples

Example terms:

- socrates
- Xiknve
- father(socrates)
- father(father(socrates))
- father(father(father(socrates)))
- son(father(socrates), Xiknve)
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supercalifragilistichespiralidoso287(A, B, C, socrates, Xiknve, D)

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Note that each term will represent an "object."

Terms Formulas

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Terms Formulas

Predicate Symbols

• Countable set *P* of predicate symbols.

• An arity is associated with each predicate symbol.

• Examples:

- human (arity 1)
- mortal (arity 1)
- married (arity 2)
- yggdrasil (arity 18)

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- If p is a predicate symbol of arity n, and
- t_1, \ldots, t_n are terms, then
- $p(t_1, \ldots, t_n)$ is an atomic formula or atom.
- Examples:
 - human(socrates)
 - mortal(father(Xiknve))
 - married(socrates, A)

Predicates with arity 0 are like propositional atoms.

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Formulas

Similar to propositional logic with different atoms.

- x is a formula if and only if
 - *x* is an atomic formula, or
 - $x = \top$, or
 - $x = \bot$, or
 - $x = (\neg y)$ where y is a formula, or
 - $x = (y \land z)$ where y, z are formulas, or
 - $x = (y \lor z)$ where y, z are formulas, or
 - $x = (y \rightarrow z)$ where y, z are formulas, or
 - $x = (y \leftrightarrow z)$ where y, z are formulas, or
 - $x = (\forall V y)$ where V is a variable and y is a formula, or
 - $x = (\exists V y)$ where V is a variable and y is a formula.

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Quantified Variables

- \forall is the universal quantifier.
- \exists is the existential quantifier.
- In $\forall V y$ and $\exists V y, y$ is the scope of V.
- In $\forall V y$ and $\exists V y$, V is bound in y.
- Variables which are not bound in a formula are free.
- Formulas without free variables are closed or sentences.

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Formulas: Examples

As for propositional logic, we minimize parentheses.

- human(socrates)
- mortal(socrates)
- $\forall A (human(A) \rightarrow mortal(A))$
- $\forall A (human(son(socrates, A)) \rightarrow married(socrates, A))$
- $\forall A ((\exists B human(son(A, B))) \rightarrow (\exists C married(A, C)))$
- $((\exists B human(son(A, B))) \rightarrow (\exists C married(A, C)))$

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Terms Formulas

Practical Example 1

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Terms Formulas

Practical Example 2

$$\begin{array}{l} \forall X \neg e(s(X), z) \\ \forall X \forall Y(e(s(X), s(Y)) \rightarrow e(X, Y)) \\ \forall X e(a(X, z), X) \\ \forall X \forall Y e(a(X, s(Y)), s(a(X, Y))) \\ \forall X e(m(X, z), z) \\ \forall X \forall Y e(m(X, s(Y)), a(m(X, Y), X)) \end{array}$$

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Definition of Free Variables

- Given a term t, the set of its free variables free(t) is defined as:
 - $free(t) = \{X\}$ if t is a variable X,
 - $free(t) = \{\}$ if t is a constant,
 - $free(t) = \bigcup_{i=1}^{n} free(t_i)$ if t is $f(t_1, \ldots, t_n)$.
- Given a formula *f*, the set of its free variables *free*(*f*) is defined as:
 - $free(f) = \bigcup_{i=1}^{n} free(t_i)$ if f is an atom $p(t_1, \ldots, t_n)$,
 - *free*(f) = {} if f is \top or \bot ,
 - free(f) = free(g) if f is $\neg g$,
 - $free(f) = free(g) \cup free(h)$ if f is $g \circ h$ for $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$,

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• $free(f) = free(g) \setminus X$ if f is $\forall X g$ or $\exists X g$

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Formulas as Trees

Every formula can be written as a formula tree: $\exists X \ p(X) \lor q(Y) \rightarrow \neg p(Y) \land r(Y)$



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Motivation Syntax Semantics **Terms Formulas**

Formulas as Trees

Every formula can be written as a formula tree: $\exists X \ p(X) \lor q(Y) \rightarrow \neg p(Y) \land r(Y)$



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Term Substitution

• Substitute term x by y in term t, denoted t[y/x]:

•
$$t[y/x] = y$$
 if $t = x$,

•
$$t[y/x] = t$$
 if $t \neq x$,

•
$$t[y/x] = f(t_1[y/x], ..., t_n[y/x])$$
 if $t \neq x$ and $t = f(t_1, ..., t_n)$.

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$$f[y/x] = \neg g[y/x]$$
 if $f = \neg g$

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$$f[y/x] = g[y/x] \circ h[y/x]$$
 if $f = g \circ h$ for $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$,

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$$f[y/x] = \mathcal{Q}V g[y/x]$$
 if $f = \mathcal{Q}V g$ for $\mathcal{Q} \in \{\forall, \exists\}$ and $V \neq x$,

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•
$$f[y/x] = p(t_1[y/x], ..., t_n[y/x])$$
 if *f* is an atom $p(t_1, ..., t_n)$

•
$$f[y/x] = \neg g[y/x]$$
 if $f = \neg g$

•
$$f[y/x] = g[y/x] \circ h[y/x]$$
 if $f = g \circ h$ for $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$,

• $f[y/x] = \mathcal{Q}V g[y/x]$ if $f = \mathcal{Q}V g$ for $\mathcal{Q} \in \{\forall, \exists\}$ and $V \neq x$,

• f[y/x] = QV g if f = QV g for $Q \in \{\forall, \exists\}$ and V = x.

Terms Formulas

Formula Substitution

- Substitute a formula g by h in a formula f, denoted f[h/g]:
- \Rightarrow As for propositional logic.

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Structures Valuation

Outline



- Why "more than" Propositional Logic?Intuition
- 2 Syntax
 - Terms
 - Formulas



Valuation

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Structures Valuation

Semantics – Requirements

Give meaning to

- function symbols (including constants),
- variable symbols
- predicate symbols
- formulas

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Structures Valuation

Domain

- We want to speak about "objects."
- Each term represents an "object."
- Domain D: A set of "objects."

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Interpretation Function

- Interpretation I: Associate terms with objects.
- For each constant symbol $c, l(c) \in D$
- For each function symbol *f* of arity *n*, *l*(*f*) is a function $\mathcal{D}^n \mapsto \mathcal{D}$
- For any *n*-tuple of objects, *l*(*f*) defines a unique object.

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Interpretation Function

- Interpretation *I*: Also associate predicates with truth functions.
- For each constant predicate *p* of arity *n*, *l*(*p*) is a function
 Dⁿ → {0, 1}
- For any *n*-tuple of objects, *I*(*p*) defines a truth value.

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Structures Valuation

Free Variables?

- There are two ways to handle free variables in formulas:
 - Define variable assignments, associating a domain object with a variable.
 - Consider universal closure: $\forall X_1 \dots \forall X_n f$ where $free(f) = \{X_1, \dots, X_n\}.$
- We will consider variable assignments:
- A variable assignment *v* is a function $V \mapsto \mathcal{D}$.

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Free Variables?

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Structures

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Structures Valuation

Outline



- Why "more than" Propositional Logic?Intuition
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Valuation

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Valuation

- A valuation μ_M of a term *t* with respect to a first order structure M = (D, I, v) is:
 - $\mu_M(t) = v(t)$ if t is a variable
 - $\mu_M(t) = I(t)$ if t is a constant
 - $\mu_M(t) = I(f)(\mu_M(t_1), ..., \mu_M(t_n))$ if t is $f(t_1, ..., t_n)$
- A valuation μ_M for a formula *f* with respect to a first order structure M = (D, I, v) is:
 - $\mu_M(\top) = 1$
 - $\mu_M(\perp) = 0$
 - $\mu_M(p(t_1,...,t_n)) = I(p)(\mu_M(t_1),...,\mu_M(t_n))$
 - continuing on next slide

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Valuation

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 - continuing on next slide

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Valuation

A valuation μ_M for a formula *f* with respect to a first order structure M = (D, I, v) is:

- $\mu_M(\neg g) = \neg \mu_M(g)$ using propositional logic
- μ_M(g ∘ h) = μ_M(g) ∘ μ_M(h) using propositional logic for ∘ ∈ {∧, ∨, →, ↔}
- $\mu_M(\forall V g) = 1$ if $\mu_M(g[d/V]) = 1$ for all $d \in \mathcal{D}$
- $\mu_M(\forall V \ g) = 0$ if $\mu_M(g[d/V]) = 0$ for some $d \in \mathcal{D}$
- $\mu_M(\exists V \ g) = 1$ if $\mu_M(g[d/V]) = 1$ for some $d \in \mathcal{D}$
- $\mu_M(\exists V \ g) = 0$ if $\mu_M(g[d/V]) = 0$ for all $d \in \mathcal{D}$

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Structures Valuation

Practical Example 1

$$\begin{array}{l} n(z) \\ \forall X \ (n(X) \rightarrow n(s(X))) \\ \forall X \ \forall Y \ (\neg(e(X,Y)) \rightarrow \neg(e(s(X),s(Y)))) \\ \forall X \ \neg e(s(X),z) \end{array}$$

- D: Natural numbers (including 0)
- I(z) = 0
- *I*(*s*): Successor of a number
- *I*(*n*): True for natural numbers
- *I*(*e*): True for equal numbers
- no free variables

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Structures Valuation





Giuseppe Peano (1858-1932)

Wolfgang Faber First-Order Logic 1

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Structures Valuation

Practical Example 2

$$\begin{array}{l} \forall X \neg e(s(X), z) \\ \forall X \forall Y(e(s(X), s(Y)) \rightarrow e(X, Y)) \\ \forall X e(a(X, z), X) \\ \forall X \forall Y e(a(X, s(Y)), s(a(X, Y))) \\ \forall X e(m(X, z), z) \\ \forall X \forall Y e(m(X, s(Y)), a(m(X, Y), X)) \end{array}$$

- D: Natural numbers (including 0)
- I(z) = 0
- *I*(*s*): Successor of a number
- *I*(*a*): Sum of two numbers
- *I*(*m*): Product of two numbers
- *I*(*e*): True for equal numbers
- no free variables

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Structures Valuation

Russell and Whitehead



Bertrand Russell (1872–1970)



Alfred Whitehead (1861-1947)

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Structures Valuation

Socrates Example

human(socrates) $\forall X(human(X) \rightarrow mortal(X))$ mortal(socrates)

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