Contents

1 Motivation

1.1 Why "more than" Propositional Logic?

Propositional Logic ⇒ Done?
We want to represent:

- Socrates is a human.
- Humans are mortal.

From this, we want to draw the conclusion:

- Socrates is mortal.

Propositional Logic ⇒ Done?
In propositional logic:

- variable SH (Socrates is a human.)
- variable HM (Humans are mortal.)
- variable SM (Socrates is mortal.)
- formula F: $SH \wedge$ ("Socrates is a human" and ...)
- $SH \rightarrow SM$ ("Socrates is a human" implies "Socrates is mortal.")

From this, we can draw the conclusion:

- $F \models SM$ ("Socrates is mortal.")
- ...and where is HM??
- This is not what we wanted to express!

Propositional Logic ⇒ Done?

"Humans are mortal" *is not an atom*!

- Marco is a human.
- Humans are mortal.
- Marco is mortal.
- We talk about "objects," not about propositions!
- *But:* There are no objects in propositional logic.

1.2 Intuition

Towards First-Order Logic

- "Start" from Propositional Logic.
- "Atoms" are no longer indivisible.
- Compose "atoms" from:
  - *terms* (representing objects) and
  - *predicates* (statements about terms).
- Socrates ⇒ term
- Mortal ⇒ predicate

Functions

- "Socrates' father"
- ... represents exactly one "object,"
- but we refer to it from another "object."
- *Function symbols* map some "objects" to another "object".
- "Objects" are constant function symbols!

Variable

- "Humans are mortal"
- ... if some "object" is human, it is mortal.
- ... We need a *variable* ranging over "objects."
- *Note:* Such variables are completely different from propositional variables!
- Think about them as *object variables*.

Quantifiers

- "Humans are mortal"
- …actually wants to express "all humans are mortal,"
- …and not "some humans are mortal,"
- …and certainly not "no humans are mortal."
- *Quantifiers* express
  1. "for all objects" or
  2. "some object exists."

2 Syntax

2.1 Terms

Function Symbols, Constants

- Countable set F of *function symbols*.
- An *arity* (nonnegative integer) is associated with each function symbol.
- Function symbols with arity 0 are *constants* ("objects").
- Examples:
  - $socrates$ (arity 0)
  - $father$ (arity 1)
  - son (arity 2)
  - $supercalifragilistichespiralidoso287$ (arity 6)

Variables

- Countable set V of (object) *variables*.
- V and F are disjoint!
- Examples:
  - $Human$
  - $Xiknve$
  - A, B, C, D

Terms

Build *terms* from function symbols and variables,respecting arities.
Again: Inductive definition.

- If v is a variable symbol, v is a term.
- If f is a function symbol of arity 0, f is a term.
- If f is a function symbol of arity $n > 0$, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Ground term: Term not containing any variable.

Terms – Examples

Example terms:

- $socrates$
- $Xiknve$
- $father(socrates)$
- $father(father(socrates))$
- $father(father(father(socrates)))$
- $son(father(socrates), Xiknve)$
- $supercalifragilistichespiralidoso287(A, B, C, socrates, Xiknve, D)$

Note that each term will represent an "object."

2.2 Formulas

Predicate Symbols

- Countable set P of *predicate symbols*.
- An *arity* is associated with each predicate symbol.
- Examples:
  - $human$ (arity 1)
  - $mortal$ (arity 1)
  - $married$ (arity 2)
  - $yggdrasil$ (arity 18)

Atoms

- If p is a predicate symbol of arity n, and
- $t_1, \ldots, t_n$ are terms, then
- $p(t_1, \ldots, t_n)$ is an atomic formula or atom.
- Examples:
  - $human(socrates)$
  - $mortal(father(Xiknve))$
  - $married(socrates, A)$

Predicates with arity 0 are like propositional atoms.

Formulas

Similar to propositional logic with different atoms.
x is a formula if and only if

- x is an atomic formula, or
- $x = \top$, or
- $x = \bot$, or
- $x = (\neg y)$ where y is a formula, or
- $x = (y \wedge z)$ where y, z are formulas, or
- $x = (y \vee z)$ where y, z are formulas, or
- $x = (y \rightarrow z)$ where y, z are formulas, or
- $x = (y \leftrightarrow z)$ where y, z are formulas, or
- $x = (\forall V\ y)$ where V is a variable and y is a formula, or
- $x = (\exists V\ y)$ where V is a variable and y is a formula.

Quantified Variables

- $\forall$ is the *universal quantifier*.
- $\exists$ is the *existential quantifier*.
- In $\forall V\ y$ and $\exists V\ y$, y is the *scope* of V.
- In $\forall V\ y$ and $\exists V\ y$, V is *bound* in y.
- Variables which are not bound in a formula are *free*.
- Formulas without free variables are *closed* or *sentences*.

Formulas: Examples

As for propositional logic, we minimize parentheses.

- $human(socrates)$
- $mortal(socrates)$
- $\forall A\ (human(A) \rightarrow mortal(A))$
- $\forall A\ (human(son(socrates, A)) \rightarrow married(socrates, A))$
- $\forall A\ ((\exists B\ human(son(A, B))) \rightarrow (\exists C\ married(A, C)))$
- $((\exists B\ human(son(A, B))) \rightarrow (\exists C\ married(A, C)))$

Practical Example 1

$$
\begin{array}{l}
n(z) \\
\forall X\ (n(X) \rightarrow n(s(X))) \\
\forall X\ \forall Y\ (\neg(e(X, Y)) \rightarrow \neg(e(s(X), s(Y)))) \\
\forall X\ \neg e(s(X), z)
\end{array}
$$

Practical Example 2

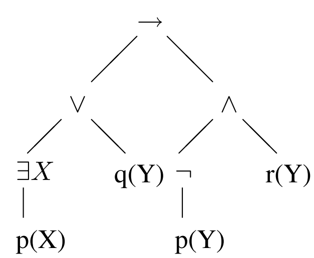
$$
\begin{array}{l}
\forall X\ \neg e(s(X), z) \\
\forall X\ \forall Y (e(s(X), s(Y)) \rightarrow e(X, Y)) \\
\forall X\ e(a(X, z), X) \\
\forall X\ \forall Y\ e(a(X, s(Y)), s(a(X, Y))) \\
\forall X\ e(m(X, z), z) \\
\forall X\ \forall Y\ e(m(X, s(Y)), a(m(X, Y), X))
\end{array}
$$

Definition of Free Variables

- Given a term t, the set of its free variables $free(t)$ is defined as:
  - $free(t) = \{X\}$ if t is a variable X,
  - $free(t) = \{\}$ if t is a constant,
  - $free(t) = \bigcup_{i=1}^{n} free(t_i)$ if t is $f(t_1, \ldots, t_n)$.
- Given a formula f, the set of its free variables $free(f)$ is defined as:
  - $free(f) = \bigcup_{i=1}^{n} free(t_i)$ if f is an atom $p(t_1, \ldots, t_n)$,
  - $free(f) = \{\}$ if f is $\top$ or $\bot$,
  - $free(f) = free(g)$ if f is $\neg g$,
  - $free(f) = free(g) \cup free(h)$ if f is $g \circ h$ for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,
  - $free(f) = free(g) \setminus X$ if f is $\forall X\ g$ or $\exists X\ g$

Formulas as Trees

Every formula can be written as a *formula tree*:
$\exists X\ p(X) \vee q(Y) \rightarrow \neg p(Y) \wedge r(Y)$



Term Substitution

- Substitute term x by y in term t, denoted $t[y/x]$:
  - $t[y/x] = y$ if $t = x$,
  - $t[y/x] = t$ if $t \neq x$,
  - $t[y/x] = f(t_1[y/x], \ldots, t_n[y/x])$ if $t \neq x$ and $t = f(t_1, \ldots, t_n)$.
- Substitute term x by y in a formula f, denoted $f[y/x]$:
  - $f[y/x] = f$ if f is $\top$ or $\bot$
  - $f[y/x] = p(t_1[y/x], \ldots, t_n[y/x])$ if f is an atom $p(t_1, \ldots, t_n)$
  - $f[y/x] = \neg g[y/x]$ if $f = \neg g$
  - $f[y/x] = g[y/x] \circ h[y/x]$ if $f = g \circ h$ for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,
  - $f[y/x] = \mathcal{Q}V\ g[y/x]$ if $f = \mathcal{Q}V\ g$ for $\mathcal{Q} \in \{\forall, \exists\}$ and $V \neq x$,
  - $f[y/x] = \mathcal{Q}V\ g$ if $f = \mathcal{Q}V\ g$ for $\mathcal{Q} \in \{\forall, \exists\}$ and $V = x$.

Formula Substitution

- Substitute a formula g by h in a formula f, denoted $f[h/g]$:
- $\Rightarrow$ As for propositional logic.

3 Semantics

3.1 Structures

Semantics – Requirements
Give meaning to

- function symbols (including constants),

- variable symbols
- predicate symbols
- formulas

Domain

- We want to speak about "objects."
- Each term represents an "object."
- *Domain* $\mathcal{D}$: A set of "objects."

Interpretation Function

- Interpretation I: Associate terms with objects.
- For each constant symbol c, $I(c) \in \mathcal{D}$
- For each function symbol f of arity n, $I(f)$ is a function $\mathcal{D}^n \mapsto \mathcal{D}$
- For any n-tuple of objects, $I(f)$ defines a unique object.

Interpretation Function

- Interpretation I: Also associate predicates with truth functions.
- For each constant predicate p of arity n, $I(p)$ is a function $\mathcal{D}^n \mapsto \{0, 1\}$
- For any n-tuple of objects, $I(p)$ defines a truth value.

Free Variables?

- There are two ways to handle *free variables* in formulas:
    1. Define *variable assignments*, associating a domain object with a variable.
    2. Consider *universal closure*: $\forall X_1 \ \ldots \forall X_n \ f$ where $free(f) = \{X_1, \ldots, X_n\}$.
- We will consider variable assignments:
- A variable assignment v is a function $V \mapsto \mathcal{D}$.

3.2 Valuation

Valuation

- A *valuation* μ_M of a term t with respect to a *first order structure* $M = (\mathcal{D}, I, v)$ is:
  - $\mu_M(t) = v(t)$ if t is a variable
  - $\mu_M(t) = I(t)$ if t is a constant
  - $\mu_M(t) = I(f)(\mu_M(t_1), \ldots, \mu_M(t_n))$ if t is $f(t_1, \ldots, t_n)$
- A *valuation* μ_M for a formula f with respect to a first order structure $M = (\mathcal{D}, I, v)$ is:
  - $\mu_M(\top) = 1$
  - $\mu_M(\bot) = 0$
  - $\mu_M(p(t_1, \ldots, t_n)) = I(p)(\mu_M(t_1), \ldots, \mu_M(t_n))$
  - continuing on next slide

Valuation

A *valuation* μ_M for a formula f with respect to a first order structure $M = (\mathcal{D}, I, v)$ is:

- $\mu_M(\neg g) = \neg\mu_M(g)$ using propositional logic
- $\mu_M(g \circ h) = \mu_M(g) \circ \mu_M(h)$ using propositional logic for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
- $\mu_M(\forall V\ g) = 1$ if $\mu_M(g[d/V]) = 1$ for all $d \in \mathcal{D}$
- $\mu_M(\forall V\ g) = 0$ if $\mu_M(g[d/V]) = 0$ for some $d \in \mathcal{D}$
- $\mu_M(\exists V\ g) = 1$ if $\mu_M(g[d/V]) = 1$ for some $d \in \mathcal{D}$
- $\mu_M(\exists V\ g) = 0$ if $\mu_M(g[d/V]) = 0$ for all $d \in \mathcal{D}$

Practical Example 1

$$
\begin{array}{l}
n(z) \\
\forall X\ (n(X) \rightarrow n(s(X))) \\
\forall X\ \forall Y\ (\neg(e(X,Y)) \rightarrow \neg(e(s(X), s(Y)))) \\
\forall X\ \neg e(s(X), z)
\end{array}
$$

- $\mathcal{D}$: Natural numbers (including 0)
- $I(z) = 0$
- $I(s)$: Successor of a number

- $I(n)$: True for natural numbers
- $I(e)$: True for equal numbers
- no free variables

Peano

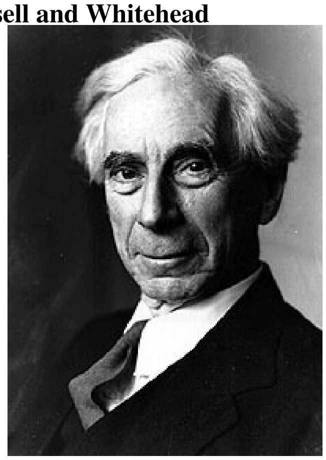
 Giuseppe Peano (1858–1932)

Practical Example 2

$$\begin{array}{l}\forall X\ \neg e(s(X), z) \\ \forall X\ \forall Y (e(s(X), s(Y)) \rightarrow e(X, Y)) \\ \forall X\ e(a(X, z), X) \\ \forall X\ \forall Y\ e(a(X, s(Y)), s(a(X, Y))) \\ \forall X\ e(m(X, z), z) \\ \forall X\ \forall Y\ e(m(X, s(Y)), a(m(X, Y), X))\end{array}$$

- $\mathcal{D}$: Natural numbers (including 0)
- $I(z) = 0$
- $I(s)$: Successor of a number
- $I(a)$: Sum of two numbers
- $I(m)$: Product of two numbers
- $I(e)$: True for equal numbers
- no free variables

Russell and Whitehead



Bertrand Russell (1872–1970)

Alfred Whitehead (1861–1947)

Socrates Example

$$human(socrates)$$
$$\forall X(human(X) \rightarrow mortal(X))$$
$$mortal(socrates)$$