Contents

1 Semantic Notions

Models

- A structure $M = (\mathcal{D}, I, v)$ is a *model* of a formula f if $\mu_M(f) = 1$
- If $\mu_M(f) = 1$, then M *satisifies* f.
- If M satisifes f, we write $M \models f$.

Satisfiability, Validity, Equivalence, Entailment

A formula f is …

- *satisfiable*, if an M exists such that $M \models f$
- *valid*, if $M \models f$ for all M.

For two formulas f and g,

- f is *equivalent* to g ($f \equiv g$), if f and g have the same models,
- f *entails* g ($f \models g$), if each model of f is also a model of g.

Socrates Example

$$
\begin{array}{l}
human(socrates) \\
\forall X(human(X) \rightarrow mortal(X)) \\
mortal(socrates)
\end{array}
$$

$$human(socrates) \wedge \forall X(human(X) \rightarrow mortal(X)) \models mortal(socrates)$$

Validity, Equivalence, Entailment as (Un)Satisfiability

- f is valid if $\neg f$ is *unsatisfiable*.
- $f \equiv g$ holds if $f \leftrightarrow g$ is valid.
- $f \equiv g$ holds if $\neg(f \leftrightarrow g)$ is unsatisfiable.
- $f \models g$ holds if $f \rightarrow g$ is valid (Deduction Theorem).
- $f \models g$ holds if $\neg(f \rightarrow g)$ is unsatisfiable.

Substitution Theorem

Theorem 1. *For wffs f, g, h, where $g \equiv h$, we obtain $f \equiv f[h/g]$*

Just as in propositional logic.

Useful Equivalences

- $f \circ g \equiv g \circ f$ (Commutativity) for $\circ \in \{\wedge, \vee, \leftrightarrow\}$
- $f \circ f \equiv f$ (Idempotence) for $\circ \in \{\wedge, \vee\}$
- $f \vee \top \equiv \top$
- $f \wedge \bot \equiv \bot$
- $f \vee \bot \equiv f$ (Neutrality)
- $f \wedge \top \equiv f$ (Neutrality)
- $f \vee \neg f \equiv \top$
- $f \wedge \neg f \equiv \bot$
- $\neg\neg f \equiv f$
- $f \rightarrow g \equiv \neg f \vee g$
- $\neg(f \vee g) \equiv \neg g \wedge \neg f$ (De Morgan)
- $\neg(f \wedge g) \equiv \neg g \vee \neg f$ (De Morgan)

Useful Equivalences 2

- $f \vee (g \vee h) \equiv (f \vee g) \vee h$ (Associativity)
- $f \wedge (g \wedge h) \equiv (f \wedge g) \wedge h$ (Associativity)
- $f \wedge (g \vee h) \equiv (f \wedge g) \vee (f \wedge h)$ (Distributivity)
- $f \vee (g \wedge h) \equiv (f \vee g) \wedge (f \vee h)$ (Distributivity)
- $f \wedge (f \vee g) \equiv f$ (Absorption)
- $f \vee (f \wedge g) \equiv f$ (Absorption)
- $(\exists X\ f) \wedge g \equiv \exists X\ (f \wedge g)$ (only if X is not free in g)
- $(\exists X\ f) \vee g \equiv \exists X\ (f \vee g)$ (only if X is not free in g)
- $(\forall X\ f) \wedge g \equiv \forall X\ (f \wedge g)$ (only if X is not free in g)
- $(\forall X\ f) \vee g \equiv \forall X\ (f \vee g)$ (only if X is not free in g)

Useful Equivalences 3

- $(\exists X\ f) \wedge (\exists X\ g) \equiv \exists X\ (f \wedge g)$
- $(\exists X\ f) \vee (\exists X\ g) \equiv \exists X\ (f \vee g)$
- $(\forall X\ f) \wedge (\forall X\ g) \equiv \forall X\ (f \wedge g)$
- $(\forall X\ f) \vee (\forall X\ g) \equiv \forall X\ (f \vee g)$
- $\neg \forall X\ f \equiv \exists X\ \neg f$ ($\forall$ De Morgan)
- $\neg \exists X\ f \equiv \forall X\ \neg f$ ($\forall$ De Morgan)
- $\forall X\ f \equiv \forall Y\ f[Y/X]$ (Renaming)
- $\exists X\ f \equiv \exists Y\ f[Y/X]$ (Renaming)
- $\forall X\ \forall Y\ f \equiv \forall Y\ \forall X f$ (Exchange)
- $\exists X\ \exists Y\ f \equiv \exists Y\ \exists X f$ (Exchange)

2 Herbrand Structures

2.1 Intuition

So Many Models!

Even for a simple formula like

$$p(c)$$

there are infinitely many structures and models.

Let us look at some of them.

So Many Models!

$$p(c)$$

Structure 1: $M_1 = (\mathcal{D}_1, I_1, \epsilon)$

- $\mathcal{D}_1 = \{a\}$
- $I_1(c) = a$
- $I_1(p)(a) = 0$
- ϵ: empty variable valuation

M_1 is not a model of this formula.

So Many Models!

$$p(c)$$

Structure 2: $M_2 = (\mathcal{D}_2, I_2, \epsilon)$

- $\mathcal{D}_2 = \{a\}$
- $I_2(c) = a$
- $I_2(p)(a) = 1$
- ϵ: empty variable valuation

M_2 is a model of this formula.

So Many Models!

$$p(c)$$

Structure 3: $M_3 = (\mathcal{D}_3, I_3, \epsilon)$

- $\mathcal{D}_3 = \{b\}$
- $I_3(c) = b$
- $I_3(p)(b) = 0$
- ϵ: empty variable valuation

M_3 is not a model of this formula.

So Many Models!

$$p(c)$$

Structure 4: $M_4 = (\mathcal{D}_4, I_4, \epsilon)$

- $\mathcal{D}_4 = \{b\}$
- $I_4(c) = b$
- $I_4(p)(b) = 1$
- ϵ: empty variable valuation

M_4 is a model of this formula.

So Many Models!

$$p(c)$$

Structure 5: $M_5 = (\mathcal{D}_5, I_5, \epsilon)$

- $\mathcal{D}_5 = \{b, c\}$
- $I_5(c) = b$
- $I_5(p)(b) = 0$
- $I_5(p)(c)$ can be 0 or 1
- ϵ: empty variable valuation

M_5 is not a model of this formula.

So Many Models!

$$p(c)$$

Structure 6: $M_6 = (\mathcal{D}_6, I_6, \epsilon)$

- $\mathcal{D}_6 = \{b, c\}$
- $I_6(c) = b$
- $I_6(p)(b) = 1$
- $I_6(p)(c)$ can be 0 or 1
- ϵ: empty variable valuation

M_6 is a model of this formula.

So Many Models!

$$p(c)$$

Structure 7: $M_7 = (\mathcal{D}_7, I_7, \epsilon)$

- $\mathcal{D}_7 = \{b, c\}$
- $I_7(c) = c$
- $I_7(p)(b)$ can be 0 or 1
- $I_7(p)(c) = 0$
- ϵ: empty variable valuation

M_7 is not a model of this formula.

So Many Models!

$$p(c)$$

Structure 8: $M_8 = (\mathcal{D}_8, I_8, \epsilon)$

- $\mathcal{D}_8 = \{b, c\}$
- $I_8(c) = c$
- $I_8(p)(b)$ can be 0 or 1
- $I_8(p)(c) = 1$
- ϵ: empty variable valuation

M_8 is a model of this formula.

So Many Models!

- All structures are quite similar!
- Changing domains does not seem to change much.
- The interpretation of predicates appears crucial.
- The interpretation of functions appears to be "isomorphic" for different domains.

Cardinality of Domain

$$p(c) \wedge \neg p(d)$$

Model: $(\mathcal{D}, I, \epsilon)$

- $\mathcal{D} = \{y, z\}$
- $I(c) = y$
- $I(d) = z$
- $I(p)(y) = 1$
- $I(p)(z) = 0$

But no model exists for any $\mathcal{D}$ with $|\mathcal{D}| < 2$!
⇒ Cardinality of the domain is important.

2.2 Main Statement

Jacques Herbrand

Jacques Herbrand (1908–1931)

Herbrand Universe

- Idea: Use the set of *ground terms* of the formula as domain!
- This domain is called *Herbrand Universe*.
- ⇒ Interpret function symbols as "themselves."
- $I_H(c) = c$ for constants
- $I_H(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$

Herbrand Universe – Example

$$\begin{array}{l} n(z) \\ \forall X\ (n(X) \rightarrow n(s(X))) \\ \forall X\ \forall Y\ (\neg(e(X,Y)) \rightarrow \neg(e(s(X),s(Y)))) \\ \forall X\ \neg e(s(X),z) \end{array}$$

- $\mathcal{D}_H = \{z, s(z), s(s(z)), s(s(s(z))), \ldots\}$
- $I_H(z) = z$
- $I_H(s)(z) = s(z)$
- $I_H(s)(s(z)) = s(s(z))$
- $I_H(s)(s(s(z))) = s(s(s(z)))$
- $\ldots$

Herbrand Base

- What about interpretations of predicate symbols?
- These are not fixed.
- Each predicate is a function from term tuples to $\{0, 1\}$.
- Write this as a set $\{p(t_1, \ldots, t_n) \mid I_H(p)(t_1, \ldots, t_n) = 1\}$
- $\Rightarrow$ The set of true ground atoms in this interpretation.
- Largest set $\{p(t_1, \ldots, t_n) \mid p$ a predicate of arity $n, t_1, \ldots, t_n$ terms$\}$ is called *Herbrand Base*.
- Denote Herbrand interpretations as subsets of the Herbrand Base.

Herbrand Base – Example

$$\begin{array}{l} n(z) \\ \forall X\ (n(X) \rightarrow n(s(X))) \\ \forall X\ \forall Y\ (\neg(e(X,Y)) \rightarrow \neg(e(s(X),s(Y)))) \\ \forall X\ \neg e(s(X),z) \end{array}$$

- $I_H(n)(z) = 1, I_H(n)(s(z)) = 1, \ldots$
- $I_H(e)(z, z) = 1, I_H(e)(z, s(z)) = 0, \ldots$
- $I_H(e)(s(z), z) = 0, I_H(e)(s(z), s(z)) = 1, \ldots$
- $I_H(e)(s(s(z)), z) = 0, I_H(e)(s(s(z)), s(z)) = 0, \ldots$
- $\ldots$

$I_H = \{n(z), n(s(z)), \ldots\} \cup \{e(z, z), e(s(z), s(z)), e(s(s(z)), s(s(z))), \ldots\}$

Herbrand Structures – Theorem

A structure for a formula with Herbrand domain (universe) and an Herbrand interpretation is an Herbrand structure.

If an Herbrand structure for a formula is a model, it is an Herbrand model.

Theorem 2. *A formula has a model if and only if it has an Herbrand model.*

Corollary 3. *A formula is satisfiable if and only if it has an Herbrand model.*

3 Normal Forms

3.1 Prenex Normal Form

Prenex Normal Form

Formulas of the following type are in *Prenex Normal Form*:

$$\mathcal{Q}_1 X_1 \ \ldots \ \mathcal{Q}_n X_n \ f$$

where

1. $\mathcal{Q}_i \in \{\forall, \exists\}$ for $1 \leq i \leq n$ and
2. f is a quantifier-free formula.

- $\mathcal{Q}_1 \ \ldots \ \mathcal{Q}_n$ is the *quantifier prefix*,
- f is the *matrix*.

Prenex Normal Form

- Move quantifiers outside ("up").
- Use the following rewritings:
  - $\neg\forall X \ f \Rightarrow_P \exists X \ \neg f$
  - $\neg\exists X \ f \Rightarrow_P \forall X \ \neg f$
  - $f \leftrightarrow g \Rightarrow_P (f \rightarrow g) \wedge (g \rightarrow f)$
  - $\mathcal{Q}X \ f \circ g \Rightarrow_P \mathcal{Q}Z1 \ (f[Z1/X] \circ g)$ $\quad Z1$ fresh, $\circ \in \{\wedge, \vee\}$
  - $\exists X \ f \rightarrow g \Rightarrow_P \forall Z1 \ (f[Z1/X] \rightarrow g)$ $\quad Z1$ fresh
  - $\forall X \ f \rightarrow g \Rightarrow_P \exists Z1 \ (f[Z1/X] \rightarrow g)$ $\quad Z1$ fresh
  - $f \circ \mathcal{Q}X \ g \Rightarrow_P \mathcal{Q}Z1 \ (f \circ g[Z1/X])$ $\quad Z1$ fresh, $\circ \in \{\wedge, \vee, \rightarrow\}$

3.2 Negation Normal Form

Negation Normal Form

- $\neg$ only in front of atomic formulas.
- At most one $\neg$ in front of atomic formulas.

Negation Normal Form

- Move negation inside ("down").
- Use the following rewritings:
  - $f \leftrightarrow g \Rightarrow_N (f \rightarrow g) \wedge (g \rightarrow f)$
  - $f \rightarrow g \Rightarrow_N \neg f \vee g$
  - $\neg \forall X\ f \Rightarrow_N \exists X\ \neg f$
  - $\neg \exists X\ f \Rightarrow_N \forall X\ \neg f$
  - $\neg(f \vee g) \Rightarrow_N \neg g \wedge \neg f$
  - $\neg(f \wedge g) \Rightarrow_N \neg g \vee \neg f$
  - $\neg\neg f \Rightarrow f$

3.3 Conjunctive Normal Form

Conjunctive Normal Form

Formulas of the following type are in *Conjunctive Normal Form*:

$$\mathcal{Q}_1 X_1\ \ldots\ \mathcal{Q}_n X_n\ \bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m_i} l)$$

where

1. $\mathcal{Q}_i \in \{\forall, \exists\}$ for $1 \leq i \leq n$ and
2. l is a literal.

- Special case of Prenex and Negation Normal Forms.

Conjunctive Normal Form

- Apply $\Rightarrow_P$ and $\Rightarrow_N$
- Then use distributivity and $\top$, $\bot$ rules:
  - $f \wedge \top \Rightarrow_C f$
  - $f \wedge \bot \Rightarrow_C \bot$
  - $f \vee \top \Rightarrow_C \top$
  - $f \vee \bot \Rightarrow_C f$
  - $f \vee (g \wedge h) \Rightarrow_C (f \vee g) \wedge (f \vee h)$

Conjunctive Normal Form

- Note: $\top$ occurs only if it is the only clause.
- Also $\bot$ occurs only if it is the only clause!

3.4 Skolemization

Thoralf Skolem

 Thoralf Skolem (1887–1963)

Skolemization

- Problem: Alternating quantifiers in CNF.
- Notation as set of clauses not directly possible.
- Introduce *Skolem functions* to eliminate one type of quantifiers!
- Here: Eliminate $\exists$.

Skolemization

$$\mathcal{Q}_1 X_1 \ \ldots \ \mathcal{Q}_n X_n \ \bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m_i} l)$$

- Work from left to right.
- Read $\forall X_1 \ \ldots \ \forall X_n \exists Y f$:
- For any combination of terms $X_1 \ \ldots \ \forall X_n$ there exists a term Y such that f holds.
- Use a new function symbol to represent that:
- Replace Y by $s(X_1, \ldots, X_n)$!

Skolemization

- Work from left to right (no arbitrary replacements)!
- $\forall X_1 \ldots \forall X_n \exists Y f \Rightarrow_S \forall X_1 \ldots \forall X_n f[s(X_1, \ldots, X_n)/Y]$,
- s must be a fresh symbol

Skolemized CNF:

$$\forall X_1 \ldots \forall X_n \bigwedge_{i=1}^{n} (\bigvee_{j=1}^{m_i} l)$$

Can be written as sets of clauses, clauses as sets of literals.

Skolemization Is Different

- $f \equiv PNF(f)$ ($PNF(f)$ Prenex Normal Form of f)
- $f \equiv NNF(f)$ ($NNF(f)$ Negation Normal Form of f)
- $f \equiv CNF(f)$ ($CNF(f)$ Conjunctive Normal Form of f)
- $f \not\equiv SCNF(f)$ ($SCNF(f)$ Skolemized Conjunctive Normal Form of f)
- Because Skolem functions can be interpreted in whatever way in models of f, which may not be a model of $SCNF(f)$ because of this.
- But: $SCNF(f) \models f$!

Skolemized CNF – Theorem

Theorem 4. *For any formula f, f is satisfiable if and only if $SCNF(f)$ is satisfiable.*

Corollary 5. *For any formula f, f is unsatisfiable if and only if $SCNF(f)$ is unsatisfiable.*