Outline

Part I: History, Syntax, Semantics

Outline of Part I

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Part II: Properties, Normal Forms, Computation

Outline of Part II

Contents

Part I
History, Syntax, Semantics

1 History

Roots of Logic

- Greece (Aristotle, Euclid of Megara [not Alexandria!])
- India (Nyaya school)
- China (Mo Zi)

all between 400BC – 100BC

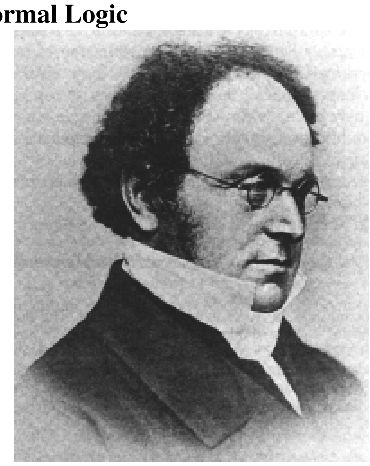
Continuation

- Arab and Islamic Logicians
- Scholastic Logic

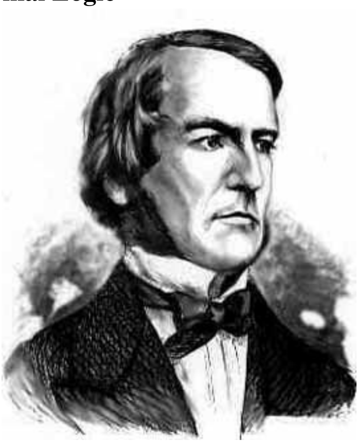
Formal Logic


Gottfried Wilhelm von Leibniz (1646–1716) ⇒ “Calculus Ratiocinator”

Formal Logic


Augustus de Morgan (1806–1871) ⇒ “Logic as Algebra”

Formal Logic


George Boole (1815–1864) ⇒ “Symbolic Logic,” “Truth Values”

2 Syntax

2.1 Intuition

Propositional Logic – Intuition

- Assume basic statements (propositions) to be given.
- Make formulas out of them using a fixed set of connectives.
- Truth of propositions determines truth of formulas.

2.2 Syntax Definition

Propositional Variables

- Countable set V of propositional variables
- Example: $\{A, B, C, D, A_0, A_1, \ldots\}$
- Important: No fixed meaning is associated to them!
- Propositional variables can mean anything.
- Truth value of variables fixed by semantics lateron.

Propositional Formulas

- Define the set F_V of propositional formulas or well-formed formulas (wff) for a set of propositional variables V.
- Inductive definition!
- A propositional variable is a wff: If $v \in V$ then $v \in F_V$.
- Propositional variables are called *atomic formulas*.
- Also: (propositional) *atoms*

Propositional Formulas

- $\top$ (*verum*) is a wff: $\top \in F_V$
- $\bot$ (*falsum*) is a wff: $\bot \in F_V$
- $\top$ and $\bot$ will have a fixed meaning.
- $\top$ is always true.
- $\bot$ is always false.

Negation

- If P is a wff, then $(\neg P)$ is a wff.
- If $P \in F_V$, then $(\neg P) \in F_V$.
- $(\neg P)$ should always have the opposite truth value of P.
- *Note:* P is a meta-symbol, and as a symbol not a wff, but a placeholder for a wff.

Conjunction

- If P and Q are wffs, then $(P \wedge Q)$ is a wff.
- If $\{P, Q\} \subseteq F_V$, then $(P \wedge Q) \in F_V$.
- $(P \wedge Q)$ should be true if both P and Q are true.
- *Note:* P and Q can be equal! P, Q are meta-symbols.

Disjunction

- If P and Q are wffs, then $(P \vee Q)$ is a wff.
- If $\{P, Q\} \subseteq F_V$, then $(P \vee Q) \in F_V$.
- $(P \vee Q)$ should be true if P, Q or both P and Q are true.
- *Inclusive* or!
- *Note:* P and Q can be equal! P, Q are meta-symbols.

Implication

- If P and Q are wffs, then $(P \rightarrow Q)$ is a wff.
- If $\{P, Q\} \subseteq F_V$, then $(P \rightarrow Q) \in F_V$.
- $(P \rightarrow Q)$ should be true if Q is true whenever P is true.
- Sometimes written as $\supset$.
- *Note:* P and Q can be equal! P, Q are meta-symbols.

Equivalence

- If P and Q are wffs, then $(P \leftrightarrow Q)$ is a wff.
- If $\{P, Q\} \subseteq F_V$, then $(P \leftrightarrow Q) \in F_V$.
- $(P \leftrightarrow Q)$ should be true if P has the same truth value as Q.
- *Note:* P and Q can be equal! P, Q are meta-symbols.

2.3 Equivalent Definitions

Compressed

$P \in F_V$ if and only if

- $P \in V$ or
- $P = \top$ or
- $P = \bot$ or
- $P = (\neg Q)$ where $Q \in F_V$ or
- $P = (Q \wedge R)$ where $Q, R \in F_V$ or
- $P = (Q \vee R)$ where $Q, R \in F_V$ or
- $P = (Q \rightarrow R)$ where $Q, R \in F_V$ or
- $P = (Q \leftrightarrow R)$ where $Q, R \in F_V$

Note: P, Q, R are meta-symbols.

Grammar

- Terminals: $V \cup \{\top, \bot\} \cup \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\} \cup \{(,)\}$
- Nonterminal: F_V
- $F_V \longrightarrow v \in V \mid \top \mid \bot$
- $F_V \longrightarrow (\neg F_V)$
- $F_V \longrightarrow (F_V \wedge F_V)$
- $F_V \longrightarrow (F_V \vee F_V)$
- $F_V \longrightarrow (F_V \rightarrow F_V)$
- $F_V \longrightarrow (F_V \leftrightarrow F_V)$

Language Elements

- V: *propositional variables* or *atoms*
- $\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$: logical *connectives*
- $(,)$: auxiliary symbols

2.4 Examples

Example Formulas

- Let $V = \{A, B, C\}$, the following are wffs:
- A
- $(A \rightarrow \bot)$
- $(A \rightarrow A)$
- $((A \vee B) \leftrightarrow (B \vee A))$
- $((A \vee B) \leftrightarrow (\neg(\top \rightarrow (A \wedge B))))$
- $((\neg(A \rightarrow B)) \rightarrow ((\neg A) \wedge C))$

Example Non-Formulas

The following are no wffs:

- $A\bot$
- $A \rightarrow \bot)$
- $A \rightarrow \neg$
- $(\rightarrow)$
- $((AB) \leftrightarrow B)$
- $((A \vee \wedge) \leftrightarrow \neg(\top)$
- $(\neg A \rightarrow B \rightarrow (\neg A \wedge C \vee B))$

2.5 Convenient Notation

Eliminating Parentheses

We can omit many (and) if we agree on a *precedence* ("binding strength") of connectives.
Usual assumption:

- $\neg$ stronger than
- $\wedge$ stronger than
- $\vee$ stronger than
- $\rightarrow$ stronger than
- $\leftrightarrow$

Examples Minimal Parentheses

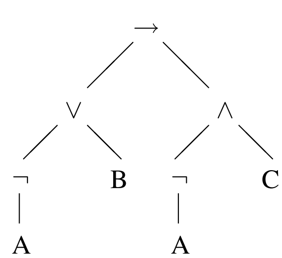
- $A \to \bot$
  $(A \to \bot)$
- $(A \to A)$
  $A \to A$
- $A \lor B \leftrightarrow B \lor A$
  $(A \lor B) \leftrightarrow (B \lor A)$
  $((A \lor B) \leftrightarrow (B \lor A))$
- $A \lor B \leftrightarrow \neg(\top \to A \land B)$
  $A \lor B \leftrightarrow (\neg(\top \to A \land B))$
  $A \lor B \leftrightarrow (\neg(\top \to (A \land B)))$
  $(A \lor B) \leftrightarrow (\neg(\top \to (A \land B)))$
  $((A \lor B) \leftrightarrow (\neg(\top \to (A \land B))))$
- $\neg A \lor B \to \neg A \land C$
  $(\neg A) \lor B \to (\neg A) \land C$
  $(\neg A) \lor B \to ((\neg A) \land C)$
  $((\neg A) \lor B) \to ((\neg A) \land C)$
  $(((\neg A) \lor B) \to ((\neg A) \land C))$

2.6 Formula Structure

Formulas as Trees

Every wff can be written as a *formula tree*:
$\neg A \lor B \to \neg A \land C$

$(((\neg A) \lor B) \to ((\neg A) \land C))$



Subformulas

- Immediate Subformula of a wff P ($isf(P)$):
  - P_0 if $P = \neg P_0$
  - P_0 if $P = P_0 \circ P_1$ for $\circ \in \{\land, \lor, \to, \leftrightarrow\}$
  - P_1 if $P = P_0 \circ P_1$ for $\circ \in \{\land, \lor, \to, \leftrightarrow\}$

- Subformula of a wff P ($sf(P)$):
  - P itself
  - If $P_s \in sf(P)$ then $isf(P_s) \subseteq sf(P)$.
  - The minimal set satisfying these conditions.

Subformulas – Example

Subformulas of $\neg A \vee B \to \neg A \wedge C$:

- $\neg A \vee B \to \neg A \wedge C$
- $\neg A \vee B$
- $\neg A \wedge C$
- $\neg A$
- B
- C
- A

3 Semantics

3.1 Meaning of a Formula

Semantics

- Associate a *meaning* to wffs in a *formal* way.
- We could associate "sentences" to variables (E.g. "It is raining.")
- But we are interested only in the *truth* or *falsity* of these sentences.
- Sentences are informal, truth values can be formalized!
- $\Rightarrow$ Associate truth values to atoms, truth values for formulas follow.

3.2 Truth Valuations

Truth valuation

- Truth values are 1 (*true*) and 0 (*false*).
- Given a set of propositional variables V,
- a (truth) *valuation* is a function ν: $V \mapsto \{0, 1\}$.
- So for each $A \in V$, either $\nu(A) = 1$ or $\nu(A) = 0$, not both.
- Now: Extend ν (for atoms) to ν^* (for wffs).

Simple Formulas

- $\nu^*(\top) = 1$
- $\nu^*(\bot) = 0$
- Always the same for any ν.
- If $A \in V$, then $\nu^*(A) = \nu(A)$.

Negation

- $\neg P$ (where P is a wff)
- $\neg P$ should always have the opposite truth value of P.
- $\nu^*(\neg P) = 1$ if (and only if) $\nu^*(P) = 0$
- $\nu^*(\neg P) = 0$ if (and only if) $\nu^*(P) = 1$

P	$\neg P$
0	1
1	0

Conjunction

- $P \wedge Q$ (where P and Q are wffs)
- $P \wedge Q$ should be true if both P and Q are true.
- $\nu^*(P \wedge Q) = 1$ if (and only if) $\nu^*(P) = 1$, $\nu^*(Q) = 1$
- $\nu^*(P \wedge Q) = 0$ if (and only if) $\nu^*(P) = 0$ or $\nu^*(Q) = 0$

P	Q	$P \wedge Q$
0	0	0
1	0	0
0	1	0
1	1	1

Disjunction

- $P \vee Q$ (where P and Q are wffs)
- $P \vee Q$ should be true if P or Q are true.
- $\nu^*(P \vee Q) = 1$ if (and only if) $\nu^*(P) = 1$ or $\nu^*(Q) = 1$
- $\nu^*(P \vee Q) = 0$ if (and only if) $\nu^*(P) = 0$, $\nu^*(Q) = 0$

P	Q	$P \vee Q$
0	0	0
1	0	1
0	1	1
1	1	1

Implication

- $P \rightarrow Q$ (where P and Q are wffs)
- $P \rightarrow Q$ should be true if Q is true whenever P is true.
- $\nu^*(P \rightarrow Q) = 1$ if (and only if) $\nu^*(P) = 1$ and $\nu^*(Q) = 1$, or if $\nu^*(P) = 0$
- $\nu^*(P \rightarrow Q) = 0$ if (and only if) $\nu^*(P) = 1$, $\nu^*(Q) = 0$

P	Q	$P \rightarrow Q$
0	0	1
1	0	0
0	1	1
1	1	1

Equivalence

- $P \leftrightarrow Q$ (where P and Q are wffs)
- $(P \leftrightarrow Q)$ should be true if P has the same truth value as Q.
- $\nu^*(P \leftrightarrow Q) = 1$ if (and only if) $\nu^*(P) = \nu^*(Q)$
- $\nu^*(P \leftrightarrow Q) = 0$ if (and only if) $\nu^*(P) \neq \nu^*(Q)$

P	Q	$P \leftrightarrow Q$
0	0	1
1	0	0
0	1	0
1	1	1

3.3 Interpretations

Interpretations

- An *interpretation* I consists exactly of a truth valuation ν.
- Given a wff P and an interpretation I consisting of valuation ν, let $I(P) = \nu^*(P)$.
- An interpretation associates a unique truth value to every formula.
- The truth value is the meaning of the formula.
- Denote as set of true variables: $\{A \mid A \in V, I(A) = 1\}$

Calculating Truth Values of Formulas

- Given a formula P and interpretation I, determine $I(P)$.
- Look at the subformulas of P.
- Work "bottom up".

Calculating Truth Values – Example

Subformulas of $\neg A \vee B \rightarrow \neg A \wedge C$:

- $\neg A \vee B \rightarrow \neg A \wedge C$, $I(\neg A \vee B \rightarrow \neg A \wedge C) = 1$
- $\neg A \vee B$, $I(\neg A \vee B) = 0$
- $\neg A \wedge C$, $I(\neg A \wedge C) = 0$
- $\neg A$, $I(\neg A) = 0$
- B, $I(B) = 0$
- C, $I(C) = 1$
- A, $I(A) = 1$

Interpretation $I = \{C, A\}$.

Calculating Truth Values

For calculating truth values for more than one interpretation, a table is useful:

A	B	C	$\neg A$	$\neg A \vee B$	$\neg A \wedge C$	$\neg A \vee B \rightarrow \neg A \wedge C$
1	0	1	0	0	0	1
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
1	0	0	0	0	0	1
0	1	1	1	1	1	1
1	1	0	0	1	0	0
1	1	1	0	1	0	0

3.4 Models

Models

- An interpretation I is a *model* of a wff P if $I(P) = 1$
- If $I(P) = 1$, then I *satisifies* P.
- If I satisifes P, we write $I \models P$.
- $I(P) = 1 \Leftrightarrow I$ is a model of $P \Leftrightarrow I$ satisifies $P \Leftrightarrow I \models P$

Non-Models

- An interpretation I is not a model of a wff P if $I(P) = 0$
- If $I(P) = 0$, then I *does not satisify* P.
- If I does not satisify P, we write $I \not\models P$.
- $I(P) = 0 \Leftrightarrow I$ is not a model of $P \Leftrightarrow I$ does not satisify $P \Leftrightarrow I \not\models P$

Part II
Properties, Normal Forms, Computation

4 Properties

4.1 Validity, Satisfiability

Validity

- A wff P is *valid* if *all* interpretations are models of P.
- A wff P is *invalid* if *not all* interpretations are models of P.
- A wff P is *satisfiable* if *there exists* an interpretation which is a model of P.
- A wff P is *unsatisfiable* if *no* interpretation is a model of P.

Classification of Formulas

A wff P is called

- *tautology* if it is *valid*
- *contradiction* if it is *unsatisfiable*.
- Tautologies are satisfiable.
- Contradictions are invalid.
- Tautologies are never contradictions.

Validity, Satisfiability – Example

A	B	C	$\neg A$	$\neg A \vee B$	$\neg A \wedge C$	$\neg A \vee B \rightarrow \neg A \wedge C$
1	0	1	0	0	0	*1*
0	0	0	1	1	0	*0*
0	0	1	1	1	1	*1*
0	1	0	1	1	0	*0*
1	0	0	0	0	0	*1*
0	1	1	1	1	1	*1*
1	1	0	0	1	0	*0*
1	1	1	0	1	0	*0*

- $\neg A \vee B \rightarrow \neg A \wedge C$ is *invalid*.
- $\neg A \vee B \rightarrow \neg A \wedge C$ is *satisfiable*.
- $\neg A \vee B \rightarrow \neg A \wedge C$ is neither a tautology nor a contradiction.

Validity, Satisfiability – Examples

- $P \rightarrow P$ tautology
- $P \wedge \neg P$ contradiction
- $P \vee \neg P$ tautology
- $P \vee \top$ tautology
- $P \wedge \bot$ contradiction

4.2 Equivalence

Equivalence

- Two wffs P, Q are *equivalent* if for each interpretation I $I(P) = I(Q)$.
- Two wffs P, Q are *equivalent* if P and Q have *the same models*.
- Two wffs P, Q are *equivalent* if $P \leftrightarrow Q$ is *valid*.
- Denote equivalence of P and Q as $P \equiv Q$.

Equivalence – Examples

- $P \lor Q \equiv Q \lor P$ (Commutativity)
- $P \land Q \equiv Q \land P$ (Commutativity)
- $P \leftrightarrow Q \equiv Q \leftrightarrow P$ (Commutativity)
- $P \lor P \equiv P$ (Idempotence)
- $P \land P \equiv P$ (Idempotence)
- $P \lor \top \equiv \top$
- $P \land \bot \equiv \bot$

Equivalence – Examples

- $P \lor \bot \equiv P$ (Neutrality)
- $P \land \top \equiv P$ (Neutrality)
- $P \lor \neg P \equiv \top$
- $P \land \neg P \equiv \bot$
- $\neg\neg P \equiv P$
- $P \rightarrow Q \equiv \neg P \lor Q$
- $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$ (Contraposition)

Equivalence – Examples

- $\neg(P \lor Q) \equiv \neg Q \land \neg P$ (De Morgan)
- $\neg(P \land Q) \equiv \neg Q \lor \neg P$ (De Morgan)
- $P \lor (Q \lor R) \equiv (P \lor Q) \lor R$ (Associativity)
- $P \land (Q \land R) \equiv (P \land Q) \land R$ (Associativity)
- $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$ (Distributivity)
- $P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R)$ (Distributivity)
- $P \land (P \lor Q) \equiv P$ (Absorption)
- $P \lor (P \land Q) \equiv P$ (Absorption)

Equivalence and Validity

- A wff P is a tautology if and only if $P \equiv \top$.
- A wff P is a contradiction if and only if $P \equiv \bot$.

4.3 Entailment

Entailment

- Interpretations could be represented as wffs!
- If $I = \{P_1, \ldots, P_n\}$, write $P_1 \wedge \ldots \wedge P_n \wedge \neg P_{n+1} \wedge \ldots \wedge \neg P_m$ where $P_{n+1}, \ldots, P_m$ are the variables which are false in I.
- Such a formula has exactly one model: I!
- So far: $I \models Q$ for interpretations I.
- $\Rightarrow$ Define $P \models Q$ also for arbitrary wffs.
- $P \models Q$ if each model of P is also a model of Q.
- If $P \models Q$, P *entails* Q.
- $P \models Q$ holds if and only if $M \models Q$ for all models M of P.
- $\models P$ if and only if P is a tautology.

Entailment – Examples

- $P \wedge Q \models P$
- $P \wedge Q \models P \rightarrow Q$
- $\neg P \models P \rightarrow Q$
- $P \models Q \rightarrow P$
- $P \models P \vee Q$

Entailment – Examples

- If $P \models Q$, then $P \wedge R \models Q$ (Monotonicity)
- $P \wedge R \models Q$ if and only if $P \models R \rightarrow Q$ (Deduction Theorem)
- $P \wedge R \models \neg Q$ if and only if $P \models Q \rightarrow \neg R$ (Contraposition Theorem)

4.4 Validity, Equivalence, Entailment as (Un)Satisfiability

Validity

- P is valid if $\top \models P$
- What about $\neg P$?
- $\neg P$ is then *unsatisfiable*.
- To check whether P is valid:
- Check whether $\neg P$ is satisfiable.
- If yes, P is not valid.
- If no, P is valid.

Equivalence

- $P \equiv Q$ holds if $P \leftrightarrow Q$ is valid.
- To check whether $P \equiv Q$ holds:
- Check whether $\neg(P \leftrightarrow Q)$ is satisfiable.
- If yes, $P \equiv Q$ does not hold
- If no, $P \equiv Q$ holds.

Entailment

- $P \models Q$ holds if $P \rightarrow Q$ is valid (Deduction Theorem).
- To check whether $P \models Q$ holds:
- Check whether $\neg(P \rightarrow Q)$ is satisfiable.
- If yes, $P \models Q$ does not hold
- If no, $P \models Q$ holds.

Formula Substitution

- For wffs P, Q, R:
- $P[R/Q]$ denotes the formula in which all occurrences of Q are replaced by R
- Example: $P_1 = (A \rightarrow B) \wedge (B \rightarrow A)$
- $Q_1 = A \rightarrow B$, $R_1 = \neg A \vee B$
- $P_1[R_1/Q_1] = (\neg A \vee B) \wedge (B \rightarrow A)$
- Example: $P_2 = (A \rightarrow B) \wedge (A \rightarrow B)$
- $Q_2 = A \rightarrow B$, $R_2 = \neg A \vee B$
- $P_2[R_2/Q_2] = (\neg A \vee B) \wedge (\neg A \vee B)$

Substitution Theorem

Theorem 1. *For wffs P, Q, R, where $Q \equiv R$, we obtain $P \equiv P[R/Q]$*

Proof. By induction over the formula structure. □

- Example: $\neg(A \wedge B) \rightarrow C$
- $\neg(A \wedge B) \equiv \neg A \vee \neg B$
- $\neg(A \wedge B) \rightarrow C \equiv (\neg A \vee \neg B) \rightarrow C$

5 Normal Forms

5.1 Why Normal Forms?

Simplify Formulas

- Many connectives: $\top$, $\bot$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- Diverse structure
- Can we find a *subset of connectives* C, such that any wff is equivalent to a formula with connectives of C?
- Can we find a *formula structure*, such that any wff is equivalent to a formula of this structure?

Eliminating Connectives

- Consider only $\neg$, $\wedge$, $\vee$!
- $\top \equiv \neg A \vee A$ (We need at least one variable for this!)
- $\bot \equiv \neg A \wedge A$ (We need at least one variable for this!)
- $P \rightarrow Q \equiv \neg P \vee Q$
- $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$
- Use repeated formula substitution to eliminate other connectives.
- Every formula is equivalent to one containing only connectives $\neg$, $\wedge$, $\vee$.

Limiting Structure

1. Conjunctive Normal Form (CNF)
   - $C_1 \wedge C_2 \wedge \ldots \wedge C_n$
   - Each *conjunct* or *clause* C_i is of the form $L_1 \vee L_2 \vee \ldots \vee L_{m_i}$.
   - Each *literal* L_{i_j} is a variable A or the negation of a variable $\neg A$.
2. Disjunctive Normal Form (DNF)
   - $D_1 \vee D_2 \vee \ldots \vee D_n$
   - Each *disjunct* D_i is of the form $L_1 \wedge L_2 \wedge \ldots \wedge L_{m_i}$.
   - Each *literal* L_{i_j} is a variable A or the negation of a variable $\neg A$.

5.2 Conjunctive Normal Form

CNF Transformation

1. Eliminate $\top$, $\bot$, $\rightarrow$, $\leftrightarrow$.
2. Apply De Morgan equivalences:
   - $\neg(P \vee Q) \equiv \neg Q \wedge \neg P$
   - $\neg(P \wedge Q) \equiv \neg Q \vee \neg P$
3. Apply double negation equivalences:
   - $\neg\neg P \equiv P$
4. Apply Distributivity equivalence:
   - $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Order of 2-4 does not matter!

Example CNF Transformation

- $\neg((A \rightarrow B) \wedge (B \leftrightarrow C))$
- $\neg((A \rightarrow B) \wedge (B \leftrightarrow C)) \equiv \neg((\neg A \vee B) \wedge (B \leftrightarrow C))$
- $\neg((\neg A \vee B) \wedge (B \leftrightarrow C)) \equiv \neg((\neg A \vee B) \wedge ((\neg B \vee C) \wedge (\neg C \vee B)))$
- $\neg((\neg A \vee B) \wedge ((\neg B \vee C) \wedge (\neg C \vee B))) \equiv \neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$
- $\neg(\neg A \vee B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv (\neg\neg A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$
- $(\neg\neg A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv (A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B))$
- $(A \wedge \neg B) \vee \neg((\neg B \vee C) \wedge (\neg C \vee B)) \equiv (A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B))$

Example CNF Transformation (2)

- $(A \wedge \neg B) \vee (\neg(\neg B \vee C) \vee \neg(\neg C \vee B)) \equiv (A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B))$
- $(A \wedge \neg B) \vee ((\neg\neg B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv (A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B))$
- $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee \neg(\neg C \vee B)) \equiv (A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B))$
- $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (\neg\neg C \wedge \neg B)) \equiv (A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B))$

Example CNF Transformation (3)

- $(A \wedge \neg B) \vee ((B \wedge \neg C) \vee (C \wedge \neg B)) \equiv (A \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(A \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (A \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(A \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$

Example CNF Transformation (4)

- $(A \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv ((A \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $((A \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge (A \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B)))$

Example CNF Transformation (5)

- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee ((B \wedge \neg C) \vee (C \wedge \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B)))$
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \wedge \neg C) \vee C) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B)))$
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \wedge \neg C) \vee \neg B))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$

Example CNF Transformation (6)

- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (\neg B \vee (((B \vee C) \wedge (\neg C \vee C)) \wedge ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge ((\neg B \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$
- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge ((\neg B \vee ((B \vee C) \wedge (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B))))$

Example CNF Transformation (7)

- $(((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge (\neg B \vee ((B \vee \neg B) \wedge (\neg C \vee \neg B)))) \equiv (((A \vee (B \vee C)) \wedge (A \vee (\neg C \vee C))) \wedge ((A \vee (B \vee \neg B)) \wedge (A \vee (\neg C \vee \neg B)))) \wedge (((\neg B \vee (B \vee C)) \wedge (\neg B \vee (\neg C \vee C))) \wedge ((\neg B \vee (B \vee \neg B)) \wedge (\neg B \vee (\neg C \vee \neg B))))$
- Flattening:
- $(A \vee B \vee C) \wedge (A \vee \neg C \vee C) \wedge (A \vee B \vee \neg B) \wedge (A \vee \neg C \vee \neg B) \wedge (\neg B \vee B \vee C) \wedge (\neg B \vee \neg C \vee C) \wedge (\neg B \vee B \vee \neg B) \wedge (\neg B \vee \neg C \vee \neg B)$

Example CNF Transformation

- $\neg((A \rightarrow B) \wedge (B \leftrightarrow C))$
- $\equiv$
- $(A \vee B \vee C) \wedge (A \vee \neg C \vee C) \wedge (A \vee B \vee \neg B) \wedge (A \vee \neg C \vee \neg B) \wedge (\neg B \vee B \vee C) \wedge (\neg B \vee \neg C \vee C) \wedge (\neg B \vee B \vee \neg B) \wedge (\neg B \vee \neg C \vee \neg B)$
- $\equiv$ (eliminate clauses containing P and $\neg P$, those are equivalent to $\top$ and are hence neutral for $\wedge$)
- $(A \vee B \vee C) \wedge (A \vee \neg C \vee \neg B) \wedge (\neg B \vee \neg C)$

CNF Representation

- $(L_{1_1} \vee L_{2_1} \vee \ldots \vee L_{m_1}) \wedge (L_{1_2} \vee L_{2_2} \vee \ldots \vee L_{m_2}) \wedge \ldots \wedge (L_{1_n} \vee L_{2_n} \vee \ldots \vee L_{m_n})$
- It is clear where which connectives are, so write it as a *set of clauses*.
- Write clauses as *sets of literals*.
- Write CNFs as a set of sets of literals:
- $\{\{L_{1_1}, L_{2_1}, \ldots, L_{m_1}\}, \{L_{1_2}, L_{2_2}, \ldots, L_{m_2}\}, \ldots, \{L_{1_n}, L_{2_n}, \ldots, L_{m_n}\}\}$

5.3 Disjunctive Normal Form

DNF Transformation

1. Eliminate $\top$, $\bot$, $\rightarrow$, $\leftrightarrow$.
2. Apply double negation equivalence:
   - $\neg\neg P \equiv P$
3. Apply De Morgan equivalences:
   - $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
   - $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
4. Apply Distributivity equivalence:
   - $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

Order of 2-4 does not matter!

6 Computation

Satisfiability: Complexity

Theorem 2 (Cook 1971)**.** *Satisfiability of wffs is NP-complete*

- First NP-completeness proof ever!
- Membership: Guess an interpretation I (from 2^n possibilities, for n variables), verify in polynomial time that $I \models \phi$.
- Hardness: Simulate nondeterministic Turing machine with polynomial time bound.

Satisfiability: Methods

- Truth Table
- DLL
- Resolution
- Tableaux
- ...

Many require CNF input!