VARIATIONAL METHODS FOR QUASILINEAR AND SEMILINEAR SINGULAR ELLIPTIC PROBLEMS

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In this PhD thesis we study problems related to the existence and the multiplicity of solutions for a class of quasilinear elliptic equations and singular semilinear elliptic equations. We consider problems in which the associated energy functional does not satisfy typical regularity conditions, in particular is not of class C^1 or even locally Lipschitz.

This situation is considered for the quasilinear elliptic equation

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times H_0^1(\Omega), \\ \int_\Omega \sum a_{ij}(x,u) D_i u D_j v \, dx + \frac{1}{2} \int_\Omega \sum D_s a_{ij}(x,u) D_i u D_j uv \, dx \\ - \int_\Omega g(x,u) v \, dx = \lambda \int_\Omega uv \, dx \qquad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and g satisfies natural growth conditions. This equation include as particular case the Schrödinger equation

$$iu_t + \triangle u - kV(x)u + k\triangle(h(\mid u \mid^2))h'(\mid u \mid^2)u + f(x,u) = 0.$$

We also consider the singular semilinear equation

$$\begin{cases} u > 0 & \text{in } \Omega\,, \\ -\Delta u = u^{-\gamma} + g(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\gamma > 0$ and g satisfies a jumping. condition.
We study the bifurcation branches of the problem (1) obtaining an extension of the Böhme-Marino theorem. A key ingredient in our proof is the nonsmooth critical point theory developed independently from M. Degiovanni and A. Ioffe.
The functional f_λ is of class C^2 on $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ but $f''_\lambda(u)$ is not a Fredholm operator. Therefore (1) can be viewed either as a problem with lack of regularity in $W_0^{1,2}(\Omega)$ or with lack of compactness in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. In any case, a standard finite-dimensional reduction of Lyapunov-Schmidt type, cannot be applied. The novelty is that we don't use a finite dimensional reduction, but we provide multiple bifurcations directly in the infinite dimensional setting. This allows weakear differentiability assumptions on a_{ij}.
We also obtain the existence of three solutions of the singular semilinear problem (2). We don't have any restriction on γ and Ω is a general domain, and we obtain solutions in the space $C(\overline{\Omega}) \cap \Big(\cap_{1 \le p < \infty} W^{2,p}_{loc}(\Omega) \Big)$.

To provide a variational setting for the problem the main nonsmooth tool that we use is a minmax theorem for functions which are C^1 perturbation of a convex and lower semicontinuous function.