Capitulation and Stabilization in various aspects of Iwasawa Theory for $\mathbb{Z}_p$-extensions

PhD Thesis – S.S.D. MAT/02, Algebra

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ABSTRACT

Let k be a number field and K/k a $\mathbb{Z}_p$-extension (where $\mathbb{Z}_p$ is the ring of p-adic integers). Consider its finite subextensions which form a tower

$$k = k_0 \subseteq k_1 \ldots \subseteq k_n \subseteq \ldots K = \bigcup_n k_n,$$

let $A(k_n)$ be the p-Sylow of the ideal class group of k_n and $X(K/k)$ the inverse limit of the $A(k_n)$ with respect to the natural norm maps. Studying the structure of $X(K/k)$ as a module over the ring $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$, Iwasawa proved, among other things, his celebrated formula (for n large enough):

$$|A(k_n)| = p^{\ \mu(K/k)p^n + \lambda(K/k)n + \nu(K/k)},$$

where $\mu(K/k)$, $\lambda(K/k)$ and $\nu(K/k)$ are constants usually called the *Iwasawa invariants* of the extension K/k.

Recently M. Ozaki has developed a nonabelian Iwasawa theory for number fields to study the maximal unramified (not necessarily abelian) pro-p extension $\widetilde{L}(K)$ of a $\mathbb{Z}_p$-extension K/k. The author breaks such a big Galois group into smaller pieces using the lower central series, which lead to the definition of higher Iwasawa modules $X^{(i)}(K/k)$ (the original Iwasawa module is $X^{(1)}(K/k)$ in this new setting). The study of $\mathrm{Gal}(\widetilde{L}(K)/K)$ relates Iwasawa theory to the problem of studying the class field tower of number fields and today there are many articles in this direction.

In the thesis we address some classical phenomenons of Iwasawa theory like capitulation and stabilization both in the abelian and nonabelian setting. The $A(k_n)$ come equipped with a natural map $i_{n,m} : A(k_n) \to A(k_m)$ (for any $m \geqslant n$) arising from the inclusion of ideals. The ideal classes in $H_{n,m} := \mathrm{Ker}(i_{n,m})$ are said to *capitulate* in k_m and those in $H_n := \bigcup_{m \geqslant n} H_{n,m}$ are said to capitulate in K. Capitulation is strictly connected to the finiteness of the Iwasawa module $X(K/k)$: indeed R. Greenberg has shown that $X(K/k)$ is finite (i.e. $\lambda(K/k) = \mu(K/k) = 0$) if and only if $A(k_n) = H_n$ for any n.

We devote the third chapter of the thesis to the classical case of $\mathbb{Z}_p$-extensions. After some preliminary results, we begin with a detailed study of the maximal finite submodule $D(K/k)$ of $X(K/k)$. Then we consider the sequences of the *absolute capitulation kernels* (the H_n) and of the *relative capitulation kernels* (the $H_{n,m}$) and we get some formulas (depending on certain parameters linked to $D(K/k)$) for their sizes, which describe accurately the growth of their orders and p-ranks. We use this description to provide some stabilization properties: in Iwasawa theory is quite usual for the order of modules like $A(k_n)$ to *stabilize* (i.e., to become constant) at the very first level $n \geqslant n_0(K/k)$ in which it does not increase if such a level exists, where $n_0(K/k)$ is the minimal layer such that every ramified prime is totally ramified, i.e.,

$$|A(k_n)| = |A(k_{n+1})| \Longrightarrow |A(k_n)| = |A(k_m)| \ \ \forall\, m \geqslant n,$$

and the same holds for the p-ranks.
We get positive answers to questions related to the stabilization of the absolute capitulation kernels: in fact the H_n enjoy a similar property while the $H_{n,m}$ do not, in general. Moreover we discuss the problem of finding bounds for the delay of capitulation and encouraged by these results we go on studying (and proving) stabilization properties for other related modules like the kernels of the norm maps or the cokernels of the inclusions.
Finally we conclude this part of the thesis by providing several new conditions (based on the properties arising from the phenomena of stabilization and capitulation) which will be shown to be equivalent to the vanishing of the Iwasawa invariants $\lambda(K/k)$ and/or $\mu(K/k)$.

We also tried to extend this type of results to the nonabelian setting and, in particular, to the higher Iwasawa modules $X^{(i)}(K/k)$ (or, more precisely, to the $X^{(i)}(k_n)$) defined by Ozaki, but we soon found an example in which the starting levels $X^{(j)}(k)$ and $X^{(j)}(k_1)$ were trivial for some $j \geqslant 2$, while $X^{(j)}(k_4) \neq 0$. This shows how the same type of stabilization (both for the orders and for the p-ranks) we had in the classical theory, is no longer valid in this new context. In fact, to obtain stabilization properties in this setting one needs more stringent hypotheses: namely stabilization for the i-th Iwasawa module requires stabilization at all the previous columns of nilpotency. More precisely,

$$\textit{if } i \geqslant 1,\ n \geqslant 0 \textit{ and } |X_n^{(j)}| = |X_{n+1}^{(j)}| \textit{ for all } 1 \leqslant j \leqslant i,$$
$$\textit{then } X_n^{(j)} \simeq X_m^{(j)} \simeq X^{(j)} \textit{ for every } 1 \leqslant j \leqslant i \textit{ and } m \geqslant n.$$

In the last chapter we consider Galois invariants and coinvariants of the $A(k_n)$. First we treat inverse and direct limits for them and study stabilization and capitulation with their consequences. Then we provide two new criterions for the finiteness of $X(K/k)$: the first is based only on the capitulation of the invariants subgroups and the second is based on the nilpotency of the Galois group $\mathrm{Gal}(L(K)/k)$ (where $L(K)$ is the maximal abelian unramified pro-p extension of K). Finally, in the last section, we show two results on the vanishing of the Iwasawa μ-invariant: in particular we study the connection between the vanishing of $\mu(F_\infty/F)$ and $\mu(kF_\infty/k)$, where F_∞/F is a $\mathbb{Z}_p$-extension of a number field F and $k \supset F$ is a Galois extension of degree p, or of degree a prime q different from p.