Introduction

This thesis is concerned with the following two problems of nonlinear analysis:

Problem 1: The estimates of the *lower Hausdorff norm* and of the *Hausdorff norm* of a retraction from the closed unit ball of an infinite dimensional Banach space onto its boundary;

Problem 2: The evaluation of the *Wosko constant* $W(X)$ for an infinite dimensional Banach space X, where $W(X)$ is the infimum of all $k \geq 1$ for which there exists a retraction of the closed unit ball onto its boundary with Hausdorff norm less than or equal to k.

Let X be a Banach space and let

$$B(X) = \{x \in X : \|x\| \leq 1\} \quad \text{and} \quad S(X) = \{x \in X : \|x\| = 1\}.$$

A continuous map $R : B(X) \to S(X)$ is a retraction if $Rx = x$, for all $x \in S(X)$
If $R : B(X) \to S(X)$ is a retraction, then $-R$ is a continuous fixed point free self-mapping of the closed unit ball. Therefore, by the Brouwer's fixed point principle, if X is finite dimensional there is no retraction from $B(X)$ onto $S(X)$.
The Scottish Book [36] contains the following question raised around 1935 by Ulam: *"Is there a retraction of the closed unit ball of an infinite dimensional Hilbert space onto its boundary?"*
In 1943, Kakutani [32] gave a positive answer to this question. Later Dugundji [21](1951) and Klee [33](1955) gave a positive answer to Ulam's question in

the more general setting of infinite dimensional Banach spaces.

Assume that X is infinite dimensional and set $\inf \emptyset = \infty$. Given a retraction $R : B(X) \to S(X)$, we denote by

$$Lip(R) = \inf\{k \geq 1 : \|Rx - Ry\| \leq k\|x - y\| \text{ for all } x, y \in B(X)\}$$

the *Lipschitz norm of* R.
In [9] Benyamini and Sternfeld, following Nowak [37], proved that for any Banach space X there is a retraction $R : B(X) \to S(X)$ with $Lip(R) < \infty$.
It is of interest in the literature the problem of evaluating the following quantitative characteristic

$$L(X) = \inf\{k \geq 1 : \text{ there is a retraction } R : B(X) \to S(X) \text{ with } Lip(R) \leq k\},$$

called the *Lipschitz constant of the space* X.
A general result states that in any Banach space X, $3 \leq L(X) \leq 256 \times 10^9$ (see [29],[2]). In special Banach spaces more precise estimates have been obtained by means of constructions depending on each space. We refer to [29] for a collection of results on this problem and related ones.
Recall that the *Hausdorff measure of noncompactness* $\gamma_X(A)$ of a bounded subset A of a Banach space X is defined by

$$\gamma_X(A) = \inf\left\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{−net in } X\right\},$$

where by a finite $\varepsilon-$net for A we mean, as usual, a finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$A \subset \bigcup_{i=1,\ldots,n} (x_i + B_\varepsilon(X))$$

with $B_\varepsilon(X) = \{x \in X : \|x\| \leq \varepsilon\}$.
Given a retraction $R : B(X) \to S(X)$, we denote by

$$\begin{aligned} \underline{\gamma}_X(R) &= \sup\{k \geq 0 : \gamma_X(R(A)) \geq k\gamma_X(A) \text{ for } A \subseteq B(X)\} \\ \text{and} & \\ \gamma_X(R) &= \inf\{k \geq 0 : \gamma_X(R(A)) \leq k\gamma_X(A) \text{ for } A \subseteq B(X)\}, \end{aligned}$$

the *lower Hausdorff norm* and the *Hausdorff norm* of R, respectively.
Actually $\underline{\gamma}_X(R)$ is a maximum, while $Lip(R)$ and $\gamma_X(R)$ are minima.
Then two interesting problems arise in nonlinear analysis (see for example [22],[41],[4],[15]) : the estimates of $\underline{\gamma}_X(R)$ and $\gamma_X(R)$ for a given retraction R and, in connection with the Hausdorff norm, the evaluation of the following quantitative characteristic

$$W(X) = \inf\{k \geq 1 : \text{ there is a retraction } R : B(X) \to S(X) \text{ with } \gamma_X(R) \leq k\},$$

called the *Wosko constant of the space X*. The constant was introduced by Wosko in [42].
From

$$\gamma_X(R(B(X))) = \gamma_X(S(X)) = \gamma_X(\overline{\text{co}}(S(X))) = \gamma_X(B(X)),$$

where $\overline{\text{co}}(X)$ is the closed convex hull of $S(X)$, it follows that $W(X) \geq 1$ for every space X.
Observe that for a given retraction R, we have $\gamma_X(R) \leq 2Lip(R)$ which becomes $\gamma_X(R) \leq Lip(R)$, when X has the ball intersection property (see [38]), infact, in this case $\gamma_X(A) = \gamma_{B(X)}(A)$ for every $A \subset B(X)$, where

$$\gamma_{B(X)}(A) = \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } B(X)\}.$$

Therefore $W(X) \leq 2L(X)$ for any space X. In particular $W(X) \leq L(X)$, if the space X has the ball intersection property.
Concerning general results about the evaluation of $W(X)$, in [41] it was proved that $W(X) \leq 6$ for any Banach space X, and $W(X) \leq 4$ for sepa-

rable or reflexive Banach spaces. It has also been proved that $W(X) \leq 3$ whenever X contains an isometric copy of l^p with $p \leq (2 - \frac{\log 3}{\log 2})^{-1} \simeq 2.41\ldots$ Moreover it has been proved that $W(X) = 1$ in some spaces of measurable functions ([14]) and in Banach spaces whose norm is monotone with respect to some basis ([4]).
Regarding Banach spaces of real continuous functions, which in the sequel we assume to be equipped with the sup norm, we cite that in $\mathcal{C}([0,1])$ for any $k > 1$, there exists a retraction $R : B(\mathcal{C}([0,1])) \to S(\mathcal{C}([0,1]))$ with $\gamma_{\mathcal{C}([0,1])}(R) \leq k$ so that $W(\mathcal{C}([0,1])) = 1$ (see [42]). However we point out that the first evaluation of the Wośko constant of the space $\mathcal{C}([0,1])$ has been given by Furi and Martelli in 1974.
In [26], [25] they have proved that $W(\mathcal{C}([0,1]))) \leq 9$.
If E is a finite dimensional normed space and K a convex compact set in E with nonempty interior, the same result has been obtained in the Banach space $\mathcal{C}(K)$ of all real continuous functions on K (see [40]), and in the Banach space $\mathcal{BC}([0,\infty))$ of all real bounded continuous functions on the noncompact interval $[0,\infty)$ (see [16]). In particular, in [16] it is shown that for any $k > 1$ the same retraction R can be chosen with $\gamma_{\mathcal{BC}([0,\infty))}(R) \leq k$ and $\underline{\gamma}_{\mathcal{BC}([0,\infty))}(R) > 0$.
The aim of this thesis is to construct retractions with *positive lower Hausdorff norms* and *small Hausdorff norms* in Banach spaces of real continuous functions which domains are not necessarily bounded or finite dimensional. Moreover by means of some examples we give explicit formulas for the *lower Hausdorff norms* and the *Hausdorff norms* of such maps.
Let S and T, with $T \subset S$, be nonempty subsets of a topological space. In the following we will denote by $\mathcal{BC}(S)$ and $\mathcal{BCU}(S)$ the Banach spaces of real all functions defined on S which are, respectively, *bounded* and *continuous*, *bounded* and *uniformly continuous*. Moreover we denote by $\mathcal{BC}_T(S)$ the Banach space of all real bounded functions that are *continuous* on S and *uniformly continuous* on T.
Assume that K is a set in a normed space E containing the closed unit ball $B(E)$. The main result of Chapter 2 is the following: For any $u > 0$ there is a

retraction $R_u : B\left(\mathcal{BC}_{B(E)}(K)\right) \rightarrow S\left(\mathcal{BC}_{B(E)}(K)\right)$ such that

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}\ (K)}(R_u) = \begin{cases} \frac{1}{2}, u \leq 4 \\ \frac{2}{u}, u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{u+8}{u}$$

In particular we have that $W(\mathcal{BC}_{B(E)}(K)) = 1.$

As corollaries we obtain similar results in the case of the space $\mathcal{BCU}(E)$ and in the space $\mathcal{BC}(E)$ when E is a finite dimensional normed space. In particular, if $E = \mathbb{R},$ we obtain [16] and, if K is a convex compact set in a finite dimensional normed space E with nonempty interior, we obtain [40]. Moreover, by the invariance of the Hausdorff measure of noncompactness under isometries, we obtain the same result in the Banach space $\mathcal{BC}_{h(B(E))}(M)$ being M a metric space homeomorphic to K under a map h, which is bilipschitzian when restricted to $B(E)$.

Chapter 3 is devoted to the construction of retractions from the closed unit ball onto the unit sphere in the Banach space $\mathcal{C}(P)$ of all real continuous functions defined on the Hilbert cube P. As in the previous chapter, we obtain explicit formulas for the *lower Hausdorff norms* and the *Hausdorff norms* of such retractions.
Our main result is the following: For any $u > 0$ there is a retraction R_u : $B(C(P))\rightarrow S(C(P))$ such that

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, \text{ if } u \leq 4 \\ \frac{4}{u}\text{ , if } u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}.$$

In particular, we have that $W(C(P)) = 1.$

Let K be a metrizable infinite dimensional compact convex set in a topological linear space and assume K is an absolute retract (e.g. a compact

convex subset of a normed space is an absolute retract). Then K is homeomorphic to the Hilbert cube ([20]), so that the Banach space $\mathcal{C}(K)$ of all real continuous functions defined on K and the space $\mathcal{C}(P)$ are isometric. Therefore, since the Hausdorff measure of noncompactness is invariant under isometries, the previous result holds also in the Banach space $\mathcal{C}(K)$.
We observe that a retraction R which has positive *lower Hausdorff norm* is a proper map, which means that the preimage $R^{-1}(M)$ of any compact set $M \subseteq X$ is compact. Thus, all the retractions we construct are proper maps. Finally we remark that the following questions are still open:

- Is $W(X) = 1$ for any infinite dimensional Banach space X?
- Is $W(X) = 1$ a minimum for any infinite dimensional Banach space X?

In [14], it is shown that the second question has a positive answer in a class of Orlicz spaces, which contains the classical Lebesgue spaces $L^p([0,1])$ $(p \geq 1)$.