Summary

This thesis is devoted to one of the classic topics about algebraic surfaces: the classification of irregular surfaces of general type and the analysis of their moduli spaces.

To a minimal surface of general type S we can associate the following numerical invariants:

- the self intersection of the canonical class K_S^2;
- the geometric genus $p_g := h^0(\omega_S)$
- the irregularity $q := h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$.

A surface S is called irregular if $q > 0$. By a theorem of Gieseker the coarse moduli space $\mathcal{M}_{a,b}$ corresponding to minimal surfaces with $K_S^2 = a$ and $p_g = b$ is a quasi projective scheme, and it has finitely many irreducible components.

The above invariants determine the other classical invariants:

- the holomorphic Euler–Poincarè characteristic $\chi(S) := \chi(\mathcal{O}_S) = 1 - q + p_g$;
- the second Chern class $c_2(S)$ of the tangent bundle which is equal to the topological Euler characteristic $e(S)$ of S.

The classical question that naturally rises at this point is the so–called geographical question, i.e., for which values of $a,\ b$ is $\mathcal{M}_{a,b}$ non-empty? The answer to this question is obviously non trivial.

There exist the following inequalities holding among the invariants of minimal surfaces of general type:

- $K_S^2, \chi \geq 1$;
- $K_S^2 \geq 2p_g - 4$ (Noether's inequality);
- if S is an irregular surface, then $K_S^2 \geq 2p_g$ (Debarre's inequality);
- $K_S^2 \leq 9\chi\mathcal{O}_S$ (Miyaoka–Yau inequality).

Thus $\chi = 1$ is the lowest possible value for a surface of general type. By the Miyaoka–Yau inequality, we have that $K_S^2 \leq 9$, hence by the Debarre's inequality we get $q = p_g \leq 4$. All known results about the classification of such surfaces are listed in [MePa, Section 2.5 a].

If $K_S^2 = 2\chi$, we have that necessarily $q = 1$. Since in this case $f : S \longrightarrow \mathrm{Alb}(S)$ is a genus 2 fibration, by using the fact that all fibres are 2–connected, the classification was completed by Catanese for $K^2 = 2$, and by Horikawa in [Hor3] in the general case.

Catanese and Ciliberto in [CaCi1] and [CaCi2] studied the case $K^2 = 2\chi + 1$, with $\chi = 1$. So in this case, by the above inequalities we get that the surfaces have the following numerical invariants:

$$K_S^2 = 3 \text{ and } p_g = q = 1.$$

The classification of such surfaces was completed by Catanese and Pignatelli in [CaPi]. The main tool for this classification is the structure theorem for genus 2 fibration, which is proved in the same work.

For $\chi \geq 2$ the situation is far more complicated and not yet studied. We consider in this thesis the case $\chi = 2$. So our surfaces have the following numerical characters

$$K_S^2 = 5,\ p_g = 2,\ q = 1.$$

By a theorem of Horikawa, which affirms that for an irregular minimal surface of general type with $2\chi \leq K^2 \leq \frac{8}{3}\chi$, the Albanese map

$$f : S \longrightarrow \mathrm{Alb}(S)$$

induces a connected fibration of curves of genus 2 over a smooth curve of genus q, we have that in the considered case a fibration $f : S \longrightarrow B$ over an elliptic curve B and with fibres of genus 2.

So we can use the results of Horikawa–Xiao and most of all those of Catanese–Pignatelli to face the challenge to completely classify all surface with the above numerical invariants. Their approach is of algebraic nature and in particular is based on a new method for studying genus 2 fibration, basically giving generators and relations of their relative canonical algebra, seen as a sheaf of algebras over the base curve B.

Our main results are as follows. First at all we studied the various possibilities for the 2–rank bundle $f_*\omega_S$. We have that $f_*\omega_S$ can be decomposable or indecomposable. In the first case the usual invariant e, associated to $f_*\omega_S$ by Xiao in [Xia1] can be equal to 0 or 2. We prove that the case $e = 2$ does not occur.

Subsequently we study the case $e = 0$ with $f_*\omega_S$ decomposable. In such case we divide the problem in various subcases. For each such subcase we study the corresponding subspace of the moduli space $\mathcal{M}$ of surfaces with $K^2 = 5$, $p_g = 2$ e $q = 1$.

By using the following formula:

$$\dim \mathcal{M} \geq 10\chi - 2K^2 + p_g = 12$$

we can consider only the strata of dimension greater than or equal to 12.

We proved that almost all the strata has dimension ≤ 11, so they don't give components of the moduli space.

The most important result is that, for the so-called strata VII, we have the following theorem.

Theorem 0.1.

(i) $\mathcal{M}_{VII,\mathrm{gen}}$ *is non-empty and of dimension* 12*;*

(ii) $\mathcal{M}_{VII2}$ *is non-empty and of dimension* 13.

References

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