PAIRINGS AND SYMMETRY NOTIONS. A NEW UNIFYING PERSPECTIVE IN MATHEMATICS AND COMPUTER SCIENCE

1. Abstract

In this thesis we introduce two types of symmetry notions, local and global, that are applicable to many different mathematical contexts. To be more specific, let Ω be a fixed set. We call a triple $\mathfrak{P} = (U, F, \Lambda)$, where U and Λ are non-empty sets and $F : U \times \Omega \to \Lambda$ is a map, a *pairing* on Ω. Then, for any subset $A \subseteq \Omega$ we define a *local symmetry relation* on U given by $u \equiv_A u'$ if $F(u, a) = F(u', a)$ for any $a \in A$, and a *global symmetry relation* on the power set $\mathcal{P}(\Omega)$ given by $A \approx_{\mathfrak{P}} A'$ if $\equiv_A$ coincides with $\equiv_{A'}$.
We use the very general notion of pairing to investigate the above two types of symmetry notions for a broad spectrum of mathematical theories very far apart, such as simple undirected graphs, digraphs, vector spaces endowed of bilinear forms, information tables, metric spaces and group actions.
The relation $\approx_{\mathfrak{P}}$ induces a closure operator $M_{\mathfrak{P}}$ on Ω and we prove that any finite lattice is order isomorphic to the closure system induced by an appropriate operator $M_{\mathfrak{P}}$. On the other hand, the relation $\approx_{\mathfrak{P}}$ also induces a set operator $C_{\mathfrak{P}}$ on Ω whose fixed point set $MINP(\mathfrak{P})$ is an abstract simplicial complex, substantially dual to the closure system induced by $M_{\mathfrak{P}}$. In this way, we obtain a closure system and an abstract simplicial complex mutually interacting by means of three set systems having relevance in both theoretical computer science and discrete mathematics. As a matter of fact, any independent set family of a matroid on Ω can be represented as the set system of the minimal partitioners of some pairing $\mathfrak{P}$ on Ω. Hence, we are enabled to investigate $MINP(\mathfrak{P})$ in relation to classical operators derived from matroid theory.
Moreover, by means of the set operator $M_{\mathfrak{P}}$, it is possible to introduce a preorder $\geq_{\mathfrak{P}}$ on $\mathcal{P}(\Omega)$ by setting $A \geq_{\mathfrak{P}} B$ if and only if $M_{\mathfrak{P}}(A) \supseteq M_{\mathfrak{P}}(B)$. This preorder (and the associated pair family $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$) satisfies the so-called *union additive property* according to which if $A'' \in \mathcal{P}(\Omega)$ and $A \geq_{\mathfrak{P}} A'$, $A \geq_{\mathfrak{P}} A''$ then $A \geq_{\mathfrak{P}} A' \cup A''$ and whose induced equivalence relation coincides exactly with $\approx_{\mathfrak{P}}$. In particular, there exists a bijection between closure operators on Ω and any preorder on $\mathcal{P}(\Omega)$ having also the above property.
Actually, the preorder $\geq_{\mathfrak{P}}$ represents a *local* version of a more general preorder on $\mathcal{P}(\mathcal{P}(\Omega)^2)$, denoted by $\rightsquigarrow_{\Omega}$, depending uniquely on the starting set Ω and mutually interrelating the preorders $\geq_{\mathfrak{P}}$, for each $\mathfrak{P} \in PAIR(\Omega)$.
In this context, the study of the preorder $\geq_{\mathfrak{P}}$ is relevant since it is possible to show that any finite lattice $\mathbb{L}$ is order-isomorphic to the symmetrization of a preorder induced by $\rightsquigarrow_{\Omega_{\mathbb{L}}}$ on the power set $\mathcal{P}(\Omega_{\mathbb{L}})$, for a suitable finite set $\Omega_{\mathbb{L}}$ and a pairing $\mathfrak{P}$ on it. On the other hand, we will see that the above preorder has an interpretation in terms of *symmetry transmission*. As a matter of fact, we define a set map $\Gamma_{\mathfrak{P}} : \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \to \mathcal{P}(U)$ and, in the finite case, an associated numerical quantity, thanks to which it is possible to evaluate the transmission of symmetry between two subsets $A, B \in \mathcal{P}(\Omega)$. In this way, the previous finite lattices representation theorem provides a refinement of order theory enabling us to connect order properties to topological, matroidal or set combinatorial properties. The epistemological consequence of all the aforementioned representation results is that we can consider

closure system, finite lattice and matroid theories as sub-theories that are parts of the more general pairing paradigm.
In such a perspective, we relate the preorder $\geq_{\mathfrak{P}}$ with the closure system $MAXP(\mathfrak{P})$ from an operatorial standpoint by considering the map $DP : \mathcal{P}(\mathcal{P}(\Omega)^2) \to \mathcal{P}(\mathcal{P}(\Omega))$ and proving that $DP(\mathcal{D}^+) = DP(\mathcal{D})$ for any $\mathcal{D} \subseteq \mathcal{P}(\mathcal{P}(\Omega)^2)$ and that $DP(\mathcal{G}(\mathfrak{P})) = DP(\mathcal{Q}(\mathfrak{P})) = MAXP(\mathfrak{P})$, where $\mathcal{G}(\mathfrak{P}) := \{(A, B) \in \mathcal{P}(\Omega)^2 : A \geq_{\mathfrak{P}} B\}$ and $\mathcal{Q}(\mathfrak{P}) := \{(X, y) : X \in MINP(\mathfrak{P}), y \in M_{\mathfrak{P}}(X) \setminus X\}$. Thus, it follows that the study of specific subset pair families $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$ is strictly related to the set systems and to the set operators characterizing pairings. This explains the need to study the main properties of the specific families satisfying the same properties as the above two previous models.
Finally, the investigation of the set map $\Gamma_{\mathfrak{P}}$ leads us to the analysis of a particular structure, namely the *indistinguishability linear systems*. In particular, we focus our attention on the corresponding concept of *compatibility* from both a local and a global standpoint. The term *compatibility* is here used in analogy with the study of the compatibility of classical finite system of linear equations with coefficients in any field $\mathbb{K}$. An indistinguishability linear system $\mathcal{S}$ is a very general structure $\langle U_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}, F_{\mathcal{S}}, \Lambda_{\mathcal{S}} \rangle$ where $U_{\mathcal{S}}$ can be seen as the equation set, $C_{\mathcal{S}}$ as the variable set, $D_{\mathcal{S}}$ consists of at most an element $d_{\mathcal{S}}$ (*constant term*) and $F_{\mathcal{S}} : U_{\mathcal{S}} \times (C_{\mathcal{S}} \cup D_{\mathcal{S}}) \to \Lambda_{\mathcal{S}}$ is a map that plays the role of the coefficients matrix in a classic linear system. Then, from a local perspective, the study of the compatibility in $\mathcal{S}$ means to fix $W \subseteq U_{\mathcal{S}}$, $A \subseteq C_{\mathcal{S}}$ and to determine in what cases all equations $u \in W$ satisfy a specific condition related to local symmetry. On the other hand, from a global perspective, we try to determine the conditions to be satisfied by some operators formalizing the compatibility notion.