Abstract PhD Thesis

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The aim of this thesis is to analyze some elliptic equations that are perturbative in nature. We will examine our problem using two tools:

(i) perturbative methods;

(ii) variational methods.

In particular we are interested in the following perturbed mixed problem

$$(\tilde{M}_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; & u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded subset of $\mathbb{R}^n$, $p \in \left(1, \frac{n+2}{n-2}\right)$, $\varepsilon > 0$ is a small parameter, and $\partial_{\mathcal{N}}\Omega$, $\partial_{\mathcal{D}}\Omega$ are two subsets of the boundary of Ω such that the union of their closures coincides with the whole $\partial\Omega$.

These problems, with mixed conditions, appear in several situations. Generally the Dirichlet condition is equivalent to impose some *state* on the physical parameter represented by u, while the Neumann conditions give a meaning at the flux parameter crossing $\partial_{\mathcal{N}}\Omega$. Here below there are some common physical applications of such problems:

- *Population dynamics.* Assume that a species lives in a bounded region Ω such that the boundary has two parts, $\partial_{\mathcal{N}}\Omega$ and $\partial_{\mathcal{D}}\Omega$, where the first one is an obstacle that blocks the pass across, while the second one is a killing zone for the population.
- *Nonlinear heat conduction.* In this case $(\tilde{M}_\varepsilon)$ models the heat (for small conductivity) in the presence of a nonlinear source in the interior of the domain, with combined isothermal and isolated regions at the boundary.
- *Reaction diffusion with semi-permeable boundary.* In this framework we have that the meaning of the Neumann part, $\partial_{\mathcal{N}}\Omega$, is an obstacle to the flux of the matter, while the Dirichlet part, $\partial_{\mathcal{D}}\Omega$, stands for a semipermeable region that allows the outwards transit of the matter produced in the interior of the cell Ω by the reaction represented by a general nonlinearity $f(u)$.

The typical concentration behavior of solutions u_ε to the above two problems is via a scaling of the variables in the form $u_\varepsilon(x) \sim U\left(\frac{x-Q}{\varepsilon}\right)$, where Q is some point of $\overline{\Omega}$, and U is a solution of

$$(1) \qquad -\Delta U + U = U^p \quad \text{in } \mathbb{R}^n \quad (\text{or in } \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}),$$

the domain depending on whether Q lies in the interior of Ω or at the boundary; in the latter case Neumann conditions are imposed. When $p < \frac{n+2}{n-2}$ (and indeed only if this inequality is satisfied), such problem (1) admits positive radial solutions which decay to zero at infinity.

Solutions of $(\tilde{M}_\varepsilon)$ which inherit this profile are called *spike layers*, since they are highly concentrated near some point of $\overline{\Omega}$.

We are interested in finding boundary spike layers for the mixed problem $(\tilde{M}_\varepsilon)$. First, we apply a perturbative approach: the idea is to obtain two compensating effects from the Neumann and the Dirichlet conditions. More precisely, calling $\mathcal{I}_\Omega$ the intersection of the closures of $\partial_{\mathcal{D}}\Omega$ and $\partial_{\mathcal{N}}\Omega$, and assuming that the gradient of H at $\mathcal{I}_\Omega$ points toward $\partial_{\mathcal{D}}\Omega$, a spike layer centered on $\partial_{\mathcal{N}}\Omega$ will be *pushed* toward $\mathcal{I}_\Omega$ by ∇H and will be *repelled* from $\mathcal{I}_\Omega$ by the Dirichlet condition.

Our main result will show that there exists a solution u_ε to the problem $(\tilde{M}_\varepsilon)$ concentrating at the interface $\mathcal{I}_\Omega$. The general strategy used relies on a finite-dimensional reduction. One finds first a manifold Z of approximate solutions to the given problem, which in our case are of the form $U\left(\frac{1}{\varepsilon}(x-Q)\right)$, and solve the equation up to a vector (in the Hilbert space) parallel to the tangent plane of this manifold. In this way one generates a new manifold $\tilde{Z}$ close to Z which represents a natural constraint for the Euler functional of $(\tilde{M}_\varepsilon)$, which is

$$\tilde{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}; \qquad u \in H^1_{\mathcal{D}}(\Omega). \tag{2}$$

Here $H^1_{\mathcal{D}}(\Omega)$ stands for the space of functions in $H^1(\Omega)$ which have zero trace on $\partial_{\mathcal{D}}\Omega$, and by *natural constraint* we mean a set for which constrained critical points of $\tilde{I}_\varepsilon$ are true critical points.

The main difficulty however is to have a good control of $\tilde{I}_\varepsilon|_{\tilde{Z}}$, which is done improving the accuracy of the functions in the original manifold Z: in fact, the better is the accuracy of these functions, the closer is $\tilde{Z}$ to Z, so the main term in the constrained functional will be given by $\tilde{I}_\varepsilon|_Z$. To find sufficiently good approximate solutions we start with those constructed in literature for the Neumann problem which reveal the role of the boundary mean curvature. However these functions are not zero on $\partial_{\mathcal{D}}\Omega$, and if one tries naively to annihilate them using cut-off functions, the corresponding error turns out to be too large. A method which revealed itself to be useful for the Dirichlet problem is to consider the *projection operator* in $H^1(\Omega)$, which consists in associating to some function in this space its closest element in $H^1_{\mathcal{D}}(\Omega)$. In our case instead, apart from having mixed conditions, the maxima of the spike-layers tend to the interface $\mathcal{I}_\Omega$, so, to better understand the projection, we need to work at a scale $d \simeq \varepsilon|\log\varepsilon|$, the order of the distance of the peak from $\mathcal{I}_\Omega$. At this scale the boundary of the domain looks nearly flat, so in this step we replace Ω with a non smooth domain $\hat{\Gamma}_D \subseteq \mathbb{R}^n$ such that part of $\partial\hat{\Gamma}_D$ looks like a cut of dimension $n-1$. We choose $\hat{\Gamma}_D$ to be even with respect to the coordinate x_n and we study H^1 projections here (with Dirichlet conditions) which are also even in x_n: as a consequence we will find functions which have zero x_n-derivative on $\{x_n = 0\} \setminus \partial\hat{\Gamma}_D$, which mimics the Neumann boundary condition on $\partial_{\mathcal{N}}\Omega$. After analyzing carefully the projection we define a family of suitable approximate solutions to $(\tilde{M}_\varepsilon)$, which turn out to have a sufficient accuracy for our analysis.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite dimensional one, and study the functional constrained on $\tilde{Z}$. If z^ε_Q denotes (roughly speaking) an approximate solution peaked at Q, with $dist(Q, \mathcal{I}_\Omega) = d_\varepsilon$, then its energy turns out to be the following

$$\tilde{I}_\varepsilon(z^\varepsilon_Q) = \varepsilon^n\left(\tilde{C}_0 - \tilde{C}_1\varepsilon H(Q) + e^{-2\frac{d_\varepsilon}{\varepsilon}(1+o(1))} + O(\varepsilon^2)\right).$$

Next, via variational methods, we analyze also the asymptotic profile of the least energy solutions to the problem $(\tilde{M}_\varepsilon)$ under generic assumptions on the domain and on the interface.

First we show that Mountain Pass solutions are in fact least energy solutions. Then we prove that, given a family of least energy solutions $\{u_\varepsilon\}$, their points of maximum must lie on the boundary of the domain Ω, as in the Neumann case.

We also analyze the rate of convergence to specify better the location of maximum limit points P_ε of the least energy solutions as $\varepsilon \to 0$: we show that the concentration point cannot belong to the interior of Dirichlet boundary part. Next, we characterize the shape of least energy solutions showing that such solutions can be approximated by the ground state solution U to the problem (1). This fact follows from other results proved in the thesis; in particular we have that, after a scaling, the maximum P_ε (indeed unique) of the solutions u_ε is always bounded away from the interface $\mathcal{I}_\Omega$ as $\varepsilon \to 0$.

Moreover, we prove that the least energy solutions concentrate at boundary points in the closure of $\partial_{\mathcal{N}}\Omega$ where the mean curvature is maximal. When this constrained maximum is attained on the interface (and if ∇H here is non zero), we will be able to show that the Mountain Pass solution has precisely the behavior found by perturbative methods.

In the last part of the thesis we consider the least energy solutions to the problem $(\tilde{M}_\varepsilon)$ and, via numerical algorithm, we construct their shape and we present the related results.

We use a numerical method which allows us to find solutions of Mountain Pass type. We consider a particular case of $(\tilde{M}_\varepsilon)$, choosing $p = 3$ and $n = 2$, namely

$$(\widetilde{\tilde{M}}_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^3 & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial_{\mathcal{N}}\Omega; \quad u = 0 \text{ on } \partial_{\mathcal{D}}\Omega; & \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^2$.

Such a problem is perturbative one with mixed boundary conditions that are numerically difficult to deal with.

We define problem $(\widetilde{\tilde{M}}_\varepsilon)$ in a bounded elliptical domain of $\mathbb{R}^2$ in order to have a non constant mean curvature H to find Mountain Pass type solutions concentrating at the interface $\mathcal{I}_\Omega$. Then, we need to *mesh* Ω in order to describe and define the *discrete differential problem* associated to $(\widetilde{\tilde{M}}_\varepsilon)$.

We want to point out that, from the numerical point of view, curved boundary domains, such as the elliptical ones, are generally more difficult to treat than the square ones.

All the algorithm, used to get the shape of least energy (Mountain Pass type) solutions of $(\widetilde{\tilde{M}}_\varepsilon)$, was implemented with a MATLAB code.