THE OPTIMAL RETRACTION PROBLEM AND SOLUTIONS OF DIFFERENTIAL SYSTEMS - ABSTRACT

LUIGI MUGLIA

In this Ph.D thesis we study the following topics related to fixed point theory:

(A) Existence of solutions for integro-differential systems with or without impulses.
(B) The optimal retraction problem and minimal displacement problem.

The contributions to the first topics appear in the following papers:

- G. Marino, P. Pietramala, L. Muglia, *Impulsive neutral integrodifferential equations on unbounded intervals*, Mediterr. J. Math. 1 1 (2004) 93–108
- G. Marino, P. Pietramala, L. Muglia, *Impulsive neutral semilinear equations on unbounded intervals*, Nonlinear Funct. Anal. Appl. 9 4 (2004) 527–543
- G. Marino, V. Colao, L. Muglia, *A note on weakly isotone maps and common solutions for differential systems*, Acta Math. Sin. (English series), 22 4 (2006) 1171-1174

The contribution to the second topics appear in the following paper:

- G. Goebel, G. Marino, L. Muglia, R. Volpe, *The retraction constant and the minimal displacement characteristic of some Banach spaces*, to appear in Nonlinear Analysis TMA

To introduce fixed point theory usually one distinguishes two principal branches: *topological theory* (TFPt) and *metric theory* (MFPt).

The first includes those topics which join topology and functional analysis (e.g. those related to Leray-Schauder theory) while, MFPt, includes methods and results that usually involve proprieties of an isometric nature.

In consequence of this we say that a set K has the *topological fixed point proprieties* (TFPp) if, for any continuous function T from K into K, there exists $x \in K$ such that $Tx = x$.

On the other hand we'll write of *metric fixed point proprieties* (MFPp) if *metric type conditions* on the map imply the existence of fixed points.

Obviously there does not exist a clean separation for this branches because metric type conditions are often used to prove theorems which are non-metric and vice-versa.

In particular, in the second chapter of this thesis, we will point out that a well-known theorem of TFPt leads a number of metric considerations on the existence of lipschitzian retractions of the ball into the spheres.

This work is set up following a tree-scheme.

We have two chapters in which we present the topics related to the fixed point theory that we studied during the Ph.D. time. Every chapter starts with an historical hint joined with the presentation of objects, technics and results which involve the topics.

The last section of any chapters contains the original results that we achieve about some problem.

The first chapter concerns the application of fixed point theory to ordinary differential equations (ODEs).

In particular we put our attention on specifies classes of equations: neutral differential equations and impulsive differential equations.

We study these classes for two reasons. Firstly because this equations are proper instruments to represent mathematical models of real phenomema like economics, physics phenomena, engineering problem et al.. Secondly because there are not many results in literature especially about strong solutions and solutions defined on unbounded intervals.

Our main results are two existence theorems for strong solutions of semilinear neutral impulsive equations and integrodifferential neutral impulsive equations, in both cases defined on unbounded intervals.

At the end of chapter 1 we insert as an appendix some results on common fixed point theory and its applications to differential systems.

Like for ODEs, searching for common solutions of differential systems can be interpreted as a search for common fixed points of opportune maps. Many authors prove by metric type conditions, similar to "contractibility" hypotheses, fixed points for condensing mapping.

In some cases these hypotheses have a consequence: the maps coincide on all of their domain. In other cases the hypotheses seem too strong to obtain results.

Starting from a recent interesting result of Dhage, we prove that, having hypotheses on the Cauchy condition of the problems joined with the weak isotonie hypothesis on the functions, gives exactly one common solution and we are able to exhibit it.

In the second chapter, citing Goebel, we present some metric consequence of topological fixed point theory. In particular we study the *minimal displacement problem* introduced by Goebel in 1973 and the *optimal retraction problem* which has its roots in the famous *Scottish Book*, problem 36 given by S. Ulam.

In the first question Goebel, by the minimal displacement constant, measures how lacks to a $k-$lipschitzian map to have fixed points. Later he defines the function $\varphi_X(k)$ to describe the sup-value of the displacements for any $k-$lipschitzian map defined from X to X, and the function $\psi_X(k)$ to describe the sup-value of the displacements for any $k-$lipschitzian map defined from the ball $B \subset X$ into itself.

For every Banach space it is known that $\psi_X(k) \leq \varphi_X(k) \leq 1 - \frac{1}{k}$. If $\psi_X(k) = 1 - \frac{1}{k}$ we say the space *extremal*.

It is an open problem (except for extremal spaces) to find a closed formula for $\psi_X(k)$ or $\varphi_X(k)$ for any Banach space.

In the optimal retraction problem one asks what is the minimal lipschitzian constant in a Banach space (that we denote by $k_0(X)$) for which there exists a $k_0(X)$-lipschitzian retraction of the ball into the sphere.

For all Banach spaces (included extremal spaces) *an exact value for* $k_0(X)$ *it is unknown.* However many authors have given some upper and lower bounds in significant spaces.

Our contribution is on this second question. We prove that for every Banach space with uniform norm that is "cut invariant" (for example $BC(I)$ with I possibly unbounded, or the space of sequences convergent to zero c_0 et al.) we have $k_0(X) \leq$

23.31. Moreover for the space of bounded functions vanishing in a point $z \in Q$ (Q a connected metric space) we obtain $k_0(BC_z(Q)) \leq 12$. In the end, following a result of Annoni and Casini ($k_0(l_1) \leq 8$) we prove that $k_0(X) \leq 8$ for the spaces $L_1[0,1]$, $AC_0[0,1]$ and $BV[0,1] \cap C_0[0,1]$.