Multiplicity of critical points for some noncoercive functionals

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Abstract: In this PhD thesis we are interested in the solvability of the quasilinear elliptic problems

$$\begin{cases} -\mathrm{div}(j_\xi(x,u,\nabla u) + j_s(x,u,\nabla u) = g(x,u) & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where $j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$j(x,s,\xi) = \frac{1}{2} a(x,s)|\xi|^2$$

under the weak assumption $a(x,s) > 0$ (that is, of degenerate coerciveness). A feature of quasilinear problems like (1) is that, for a general $u \in H^1_0(\Omega)$, the term $j_s(x,u,\nabla u)$ belongs to $L^1(\Omega)$ under reasonable assumptions, but not to $H^{-1}(\Omega)$. As a consequence, the functional

$$f(u) = \int_\Omega j(x,u,\nabla u) - \int_\Omega G(x,u)\,, \qquad G(x,s) = \int_0^s g(x,t)\,dt\,,$$

whose Euler-Lagrange equation is represented by (1), is continuous on $H^1_0(\Omega)$, but not locally Lipschitz, in particular not of class C^1. The main purpose of this thesis is to prove that, with a symmetry assumption, problem (1) possesses infinitely many solutions. We show that, by a change of variable, the degenerate case can be reduced to a suitable form that allows us to the results of the nonsmooth critical point theory developed from M. Degiovanni and A. Ioffe.