Total and Convex Total Boundedness in F-seminormed Spaces. Implicit Functions in Locally Convex Linear Spaces.

Abstract PhD Thesis

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This thesis consists of two independent parts.
The first is concerned with the following two problems of Non Linear Functional Analysis:

(A) The estimate of the measure of non convex total boundedness in terms of simpler quantitative characteristics in the space L_0 of measurable functions;

(B) The comparison of the measure of non equiabsolute continuity with the Hausdorff noncompactness measure and the non convex total boundedness measure in concrete function spaces.

The second part deals with the study of the following topic of Non Linear Operator Theory:

(C) Implicit and inverse function theorems.

The *Scottish Book* contains the following question [*Problem 54*] raised around 1935 by Schauder: *Does every compact, convex subset of a Hausdorff topological linear space have the fixed point property?* In 1987 Idzik gave the following definition:
A subset M of a topological linear space X is said to be convexly totally bounded (ctb for short) if for every 0-neighborhood U there are points $f_1, ..., f_n \in M$ and a finite number of convex subsets $C_1, ..., C_n$ of U such that

$$M \subseteq \bigcup_{i=1}^{n} (f_i + C_i) .$$

Further,he proved that *Every compact, convex and ctb subset of a Hausdorff topological linear space has the fixed point property.*
If X is locally convex every totally bounded subset of X is ctb. This is not true, in general, if X is nonlocally convex.
In 1993 De Pascale, Trombetta and Weber defined the measure of nonconvex total boundedness, modelled on Idzik's concept, that may be regarded as the analogue of the well-known notion of Hausdorff measure of noncompactness in nonlocally convex linear spaces.
Let $(G, \| \cdot \|_G)$ be a normed group, Ω a non empty set and $\mathcal{P}(\Omega)$ the power set of Ω.
The space $L_0 := L_0(\Omega, \mathcal{A}, G, \eta)$ is a space of $G-$valued functions defined on Ω which depends on an algebra $\mathcal{A}$ in $\mathcal{P}(\Omega)$ and a submeasure $\eta : \mathcal{P}(\Omega) \longrightarrow [0, +\infty]$. In particular, if $(G, \| \cdot \|_G) = (E, \| \cdot \|_E)$ is a Banach space, $\mu : \mathcal{A} \longrightarrow \overline{\mathbb{R}}$ is a finitely additive measure and η

its total variation then the space $L_0 = L_0(\mathcal{A}, \Omega, E, \eta)$ coincides with the space of measurable functions introduced by Dunford and Schwartz, in order to develop the integration theory with respect to finitely additive measures.
In Chapter 2, under the hypothesis that $(G, \|\cdot\|_G) = (E, \|\cdot\|_E)$ is a normed space, we estimate the measure of nonconvex total boundedness in $L_0(\mathcal{A}, \Omega, E, \eta)$ and we characterize the convexly totally bounded subsets of L_0. For a subset M of L_0 we introduce two quantitative characteristics $\lambda_{0,w}(M)$ and $\omega_{0,w}(M)$ involving convex sets, which measure, respectively, the *degrees of nonconvex equal-quasi boundedness* and *of nonconvex equal-measurability* of M. Then, let $\gamma_{0,w}(M)$ be the measure of non convex total boundedness of M, we establish some inequalities between $\gamma_{0,w}(M)$, $\lambda_{0,w}(M)$ and $\omega_{0,w}(M)$ that give, as a special case, a Fréchet-Šmulian type convex total boundedness criterion in the space L_0. Moreover, we extend our result to the space $\mathcal{L}_0$ that is a generalization of the space L_0.
In many concrete situations one would like to have an estimate of the degree of non compactness of a subset of a metric space. This for example happens, with problems in the theory of functional equations, including ordinary equations, equations with partial derivatives, integral and integro-differential equation, in the optimal control theory, fixed point theory, approximation theory and geometric theory of Banach spaces.
For this various authors introduced other quantitative characteristics and they compared such quantitative characteristics with the classical Hausdorff or Kuratowski measures of noncompactness. By these comparisons they obtained some inequalities, that give, as a special case the classical compactness criteria of Arzelà-Ascoli, Fréchet-Šmulian and Vitali.
In Chapter 3, under the hypothesis that $(G, \|\cdot\|_G) = (E, \|\cdot\|_E)$ is an $F-$normed, we consider a class $\mathcal{C}$ of $F-$seminormed subspaces of the space E^Ω of all $E-$valued functions defined on Ω; and for $L \in \mathcal{C}$, we introduce a parameter Π_L which measure the degree of *nonequiabsolute continuity* of subsets of L. Then we compare the above quantitative characteristic with the Hausdorff noncompactness measure and with the non convex total boundedness measure. By these comparisons we establish some inequalities. In particular, as a special case of these inequalities, we get sufficient conditions for the total boundedness of a set of functions and for the convex total boundedness of a convex set of functions. Moreover, in special spaces $L \in \mathcal{C}$, for example in the vector-valued Orlicz's spaces, from our results we derive a Vitali-type compactness criterion and a convex total boundedness criterion.
Implicit function theorems are an important tool in nonlinear analysis. They have significant applications in the theory of nonlinear integral equations. One of the most important results is the classic Hildebrandt-Graves theorem. The main assumption in all its formulations is some differentiability requirement. Applying this theorem to various types of Hammerstein integral equations in Banach spaces, it turned out that the hypothesis of existence and continuity of the derivative of the operators related to the studied equation is too restrictive.
In 1969 Zabrejko, Kolesov e Krasnosel'skij have introduced the following interesting linearization property for parameter dependent operators in Banach spaces.
Let $X = (X, \|\cdot\|_X)$ and $Y = (Y, \|\cdot\|_Y)$ be Banach spaces, Λ an open subset of the real line $\mathbb{R}$ or of the complex plane $\mathbb{C}$, A an open subset of the product space $\Lambda \times X$ and $\mathcal{L}(X, Y)$ the space

of all continuous linear operators from X into Y. An operator $\Phi : A \longrightarrow Y$ and an operator function $L : \Lambda \longrightarrow \mathcal{L}(X,Y)$ are called *osculating* at $(\lambda_0, x_0) \in A$ if there exists a function $\sigma : \mathbb{R}^2 \to [0, +\infty)$ such that

$$\lim_{(\rho,\, r)\to(0,0)} \sigma(\rho, r) = 0$$

and

$$\|\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)\|_Y \leq \sigma(\rho, r)\, \|x_1 - x_2\|_X \,,$$

when $|\lambda - \lambda_0| \leq \rho$ and $\|x_1 - x_0\|_X$, $\|x_2 - x_0\|_X \leq r$.
In Capter 4 we prove an implicit function theorem in locally convex topological linear spaces, where the classical conditions of differentiability, are replaced by the above linearization property, suitably reformulated. Moreover, as an example of application, we study the stability of the solutions of an Hammerstein equation depending on a parameter.