Introduction to Stochastic Programming

PhD Course

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- SP is a branch of mathematical programming widely used to deal with decision problems under uncertainty
- SP involves an artful blend of traditional (deterministic) mathematical programs and stochastic models
- Operations Research, Probability Calculus and Statistics are the main building blocks

Outline

The course is organized in three main lectures

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2. Main paradigms:

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2. Main paradigms:
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   - Probabilistic Constraints
3. Selected applications

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  - ...
- How do we take decisions under uncertainty ?
- **Stochastic Programming is a way !**

- Let us consider a deterministic linear programming problem:

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- In many practically relevant cases, however, problem data may be subject to uncertainty
- We assume that uncertain data can be represented as **random variables defined on a given probability space** $(\Omega, \Im, \mathbb{P})$

Stochastic LP models

- The previous model becomes

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\begin{aligned}
\text{"min"} \quad & c^T(\omega)x \\
& T(\omega)x \geq h(\omega) \ \forall \omega \in \Omega \\
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- Dealing with uncertainty makes life much harder:
- We have to worry about optimality
  - What does minimizing a random function mean ?
    We could consider the expected value or the variance
- We should worry about solution feasibility
  - It may occur that a feasible solution in all possible cases does not exist or that it is too expensive!!

An example

Let us consider the following problem

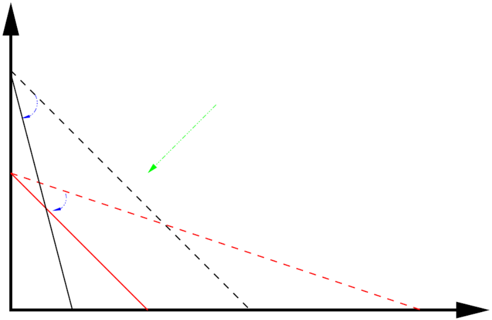
$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
& \omega_1 x_1 + x_2 \geq 7 \\
& \omega_2 x_1 + x_2 \geq 4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\omega_1 &\sim U(1,4) \\
\omega_2 &\sim U(\tfrac{1}{3},1)
\end{aligned}
$$

What to do ?

$$\omega_1 x_1 + x_2 \geq 7 \qquad \omega_1 \sim U(1,4)$$

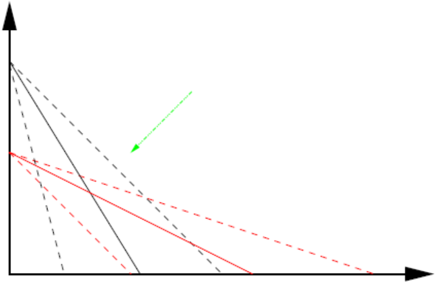
$$\omega_2 x_1 + x_2 \geq 4 \qquad \omega_2 \sim U(\frac{1}{3},1)$$



Take the mean of each random variable

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
& \frac{5}{2} x_1 + x_2 \geq 7 \\
& \frac{2}{3} x_1 + x_2 \geq 4 \\
& x_1 , x_2 \geq 0
\end{aligned}
$$

$x_1^* = 18/11 \quad x_2^* = 32/11$



Remark

- The optimal solution associated with the mean value is not feasible for the problem associated with $\omega_1 = 1$ and $\omega_2 = 1/3$

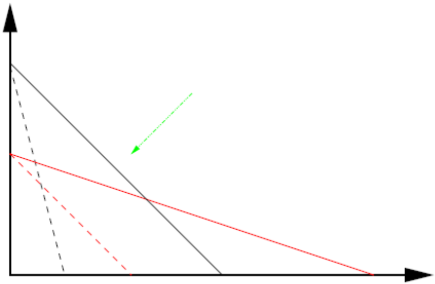
$$x_1 + x_2 \geq 7$$

$$\frac{18}{11} + \frac{32}{11} \leq 7$$

Pessimistic

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
& 1x_1 + x_2 \geq 7 \\
& \color{red}{\frac{1}{3}x_1 + x_2 \geq 4} \\
& x_1, x_2 \geq 0
\end{aligned}
$$

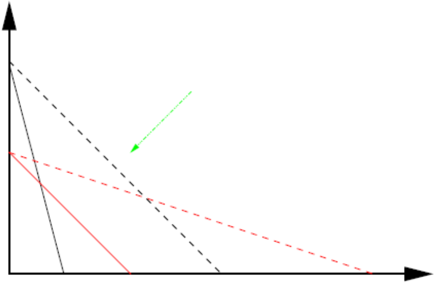
$$x_1^* = 0 \quad x_2^* = 7$$



Optimistic

$$\begin{aligned} \min \quad & x_1 + x_2 \\ & 4x_1 + x_2 \geq 7 \\ & \color{red}{1x_1 + x_2 \geq 4} \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$x_1^* = 4 \quad x_2^* = 0$$



Remark

- The optimal solution associated with the scenario $(4, 1)$ is not feasible for the problem associated with the scenario $(1, \frac{1}{3})$

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$$x_1^* = 4 \quad x_2^* = 0$$

$$1x_1 + x_2 \geq 7$$
$$\frac{1}{3}x_1 + x_2 \geq 4$$

$$4 + 0 \not\geq 7$$
$$\frac{4}{3} + 0 \not\geq 4$$

Pros and Cons of the previous approaches

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+ Easy
  We solve a deterministic problem of the same size of the original one
+ Only rough information about the randomness is needed
- Only takes into account one "case" of what randomness might be
- There might even be ω values for which the chosen optimal solution x^* is infeasible

Main approaches

- To cope with the issue of solution feasibility, two main paradigms have been proposed:

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  - **Chance Constraints**
    The optimal solution is feasible with high probability
  - **Recourse**
    We accept infeasibility, but we control the average violation (simple recourse)

- We enforce that the stochastic constraints are satisfied with a prescribed probability level α chosen by the decision maker

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$$\mathbb{P}(\omega_1 x_1 + x_2 \geq 7) \geq \alpha$$
$$\mathbb{P}(\omega_2 x_1 + x_2 \geq 4) \geq \alpha$$

Chance Constraints Approach

- We enforce that the stochastic constraints are satisfied with a prescribed probability level α chosen by the decision maker

$$\mathbb{P}(\omega_1 x_1 + x_2 \geq 7) \geq \alpha$$
$$\mathbb{P}(\omega_2 x_1 + x_2 \geq 4) \geq \alpha$$

- Chance-constrained models are very useful in some situations (reliability of the system), but they are very difficulty to deal with because they are in general not convex

Recourse Approach

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$$\omega_1 x_1 + x_2 + y_1(\omega_1) \geq 7$$

$$\omega_2 x_1 + x_2 + y_2(\omega_2) \geq 4$$

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- $y_1(\omega_1)$ and $y_2(\omega_2)$ will be exactly the shortfall in the two constraints

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- $y_1(\omega_1)$ and $y_2(\omega_2)$ will be exactly the shortfall in the two constraints
- By denoting by q_1 and q_2 the costs associated with the two new variables the objective function becomes

$$\min \; x_1 + x_2 + \mathbb{E}_\omega[q_1 y_1(\omega_1) + q_2 y_2(\omega_2)]$$

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- General recourse
  We gain feasibility after a correction has been made

Example: Farmer Ted

- Farmer Ted can grow Wheat, Corn, or Beet
- Farmer Ted requires 200 tons of wheat and 240 tons of corn to feed his cattle
- These can be grown on his land or bought from a wholesaler.
- Any production in excess of these amounts can be sold for \$170/ton (wheat) and \$150/ton (corn)
- Any shortfall must be bought from the wholesaler at a cost of \$238/ton (wheat) and \$210/ton (corn).
- Farmer Ted can also grow beets
- Beets can be sold at \$36/ton for the first 6000 tons
- Due to economic quotas on beet production, beets in excess of 6000 tons can only be sold at \$10/ton

Farmer Ted

- Farmer Ted has 500 acres available for planting
- The planting unitary costs are 150 for wheat, 230 for corns and 260 for beets, respectively
- On the basis of his experience he can evaluate the Yield of each plantation
  (T/acre) 2.5 for wheat, 3 for corn and 20 for beets

Formulate the LP- Decision Variables

- x_i Acres of Wheat (1), Corn(2), Beets(3) planted
- y_i Tons of Wheat, Corn purchased $i = 1, 2$
- w_1 Tons of wheat sold
- w_2 Tons of corn sold
- w_3 Tons of beets sold at favorable price
- w_4 Tons of beets sold at lower price

Deterministic Formulation

$$\max z = -150x_1 - 230x_2 - 260x_3 -$$
$$238y_1 + 170w_1 - 210y_2 + 150w_2 + 36w_3 + 10w_4$$

$$x_1 + x_2 + x_3 \leq 500$$
$$2.5x_1 + y_1 - w_1 \geq 200$$
$$3x_2 + y_2 - w_2 \geq 240$$
$$20x_3 - w_3 - w_4 \geq 0$$
$$w_3 \leq 6000$$
$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0$$

Solution of the deterministic LP

	WHEAT	CORN	BEETS
Plant (acres)	120	80	300
Production	300	240	6000
Sales	100	0	6000
Purchase	0	0	0

- Profit: \$118,600

Planning intuition

- Farmer Ted is happy to see that the LP solution corresponds to his intuition.
- Plant the land necessary to grow up to his quota limit of beets.
- Plant land necessary to meet his requirements for wheat and corn
- Plant remaining land with wheat sell excess.

Planning intuition

- Farmer Ted knows well enough to know that his yields aren't always precisely $Y = (2.5; 3; 20)$.
- He decides to run two more scenarios (+20 %; -20 %)
  - Good weather: 1.2Y
  - Bad weather: 0.8Y

Formulation - good yield

$$\max z = -150x_1 - 230x_2 - 260x_3 -$$
$$238y_1 + 170w_1 - 210y_2 + 150w_2 + 36w_3 + 10w_4$$

$$x_1 + x_2 + x_3 \leq 500$$
$$3x_1 + y_1 - w_1 \geq 200$$
$$3.6x_2 + y_2 - w_2 \geq 240$$
$$24x_3 - w_3 - w_4 \geq 0$$
$$w_3 \leq 6000$$
$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0$$

Solution with good yields

	WHEAT	CORN	BEETS
Plant (acres)	183.33	66.67	250
Production	550	240	6000
Sales	350	0	6000
Purchase	0	0	0

- Profit: \$167,667

Formulation - bad yield

$$\max z = -150x_1 - 230x_2 - 260x_3 -$$
$$238y_1 + 170w_1 - 210y_2 + 150w_2 + 36w_3 + 10w_4$$

$$x_1 + x_2 + x_3 \leq 500$$
$$2x_1 + y_1 - w_1 \geq 200$$
$$2.4x_2 + y_2 - w_2 \geq 240$$
$$16x_3 - w_3 - w_4 \geq 0$$
$$w_3 \leq 6000$$
$$x_1, x_2, x_3, y_1, y_2, w_1, w_2, w_3, w_4 \geq 0$$

Solution with bad yields

	WHEAT	CORN	BEETS
Plant (acres)	100	25	375
Production	200	60	6000
Sales	0	0	6000
Purchase	0	180	0

- Profit: \$59,950

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- Idea: include all the scenarios in the same formulation
- Observe that the decisions refer two different temporal moments:
- Plant decision should be made here-and-now
- Sell and purchase decision can be postponed.
- Farmer Ted has *recourse*. After he observes the weather event, he can decide how much of each crop to sell or purchase !

Formulation

- Assume that the three scenarios occur with equal probability.
- Attach a scenario subscript s = 1, 2, 3 to each of the purchase and sale variables.
- s=1 Good, s=2 Average, s=3 Bad
- Thus, for example,
  w_{1s} Tons of corn sold under scenario s
  y_{2s} Tons of corn purchased under scenario s

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$$w_{31} \leq 6000$$
$$2.5x_1 + y_{12} - w_{12} \geq 200$$
$$3x_2 + y_{22} - w_{22} \geq 240$$
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$$w_{32} \leq 6000$$

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$$16x_3 - w_{33} - w_{43} \geq 0$$
$$w_{33} \leq 6000$$

Maximize the Expected profit

$$
\begin{aligned}
\max z = & -150x_1 - 230x_2 - 260x_3 \\
& +\tfrac{1}{3}(-238y_{11} + 170w_{11} - 210y_{21} + 150w_{21} + 36w_{31} + 10w_{41}) \\
& +\tfrac{1}{3}(-238y_{12} + 170w_{12} - 210y_{22} + 150w_{22} + 36w_{32} + 10w_{42}) \\
& +\tfrac{1}{3}(-238y_{13} + 170w_{13} - 210y_{23} + 150w_{23} + 36w_{33} + 10w_{43})
\end{aligned}
$$

Optimal solution of the stochastic formulation

		Wheat	Corn	Beets
s	Plant (acres)	170	80	250
1	Production	510	288	6000
1	Sales	310	48	6000
1	Purchase	0	0	0
2	Production	425	240	5000
2	Sales	225	0	5000
2	Purchase	0	0	0
3	Production	340	192	4000
3	Sales	140	0	4000
3	Purchase	0	48	0

- (Expected) Profit: $108,390

Remarks

- Suppose Farmer Ted could with certainty tell whether or not the upcoming growing season was going to have good yields, average yields, or bad yields.
- The real point here is how much Farmer Fred would be willing to pay for this perfect information.
- In real-life problems, how much is it worth to invest in better forecasting technology?

What's it worth ?

- With perfect information, Farmer Ted would plant (wheat, corn, beans).
  - Good yield: (183.33, 66.67, 250), Profit: \$167,667
  - Average yield: (120, 80, 300), Profit: \$118,600
  - Bad yield: (100, 25, 375), Profit: \$59,950
- Assuming each of these scenarios occurs with probability 1/3, his long run average profit would be

$$(1/3)(167667) + (1/3)(118600) + (1/3)(59950) = 115406$$

- With his (optimal) here-and-now decision of (170, 80, 250), he would make a long run profit of 108390
- This difference (115406-108390) is the expected value of perfect information (EVPI)
- It represents how much farmer Ted is willing to pay to have perfect information