Introduction to Stochastic Programming

PhD Course

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Two-stage models with recourse

As we have seen in the farmer Ted example, there are some situations in which the decision process can be divided in stages: some of the decisions must be made "'here"' and "'now"' before knowing the realization of the random parameters, while others can be delayed afterward.

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1. We make a decision now (first-stage decisions)
2. Nature makes a random decision
3. We make a second decision that attempts to repair the havoc procured by nature (second-stage decisions)

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- The second stage decisions result in a cost $q(\omega)^T y(\omega)$.
- We want to minimize the sum of the first-stage cost and the expected value of second-stage cost.

Two-stage models with recourse

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\min \quad & c^T x + \mathbb{E}_\omega[q^T(\omega)y(\omega)] \\
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- $W(\omega)$ recourse matrix
- If $W(\omega)$ does not change with ω we have a **fixed recourse**
- If $W = (I - I)$ where I denotes the identity matrix we have a **fixed simple recourse**

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- We focus on the case of discrete distributions
- They arise either naturally or as approximation of the continuous case

Two-stage models with recourse: the discrete case

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- We assume that the random parameters take a given number S of realizations (scenarios) each occurring with a given probability p_s

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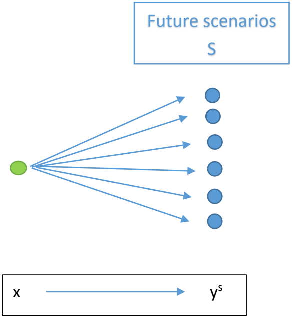
- Scenarios are represented by the following scenario tree (fan)
- First-stage decisions x must be made here and now, at the root of the tree
- Second-stage decisions y^s are scenario-dependent



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- $(m_1 + S \times m_2)$ rows and $(n_1 + S \times n_2)$ columns
- If the decision variables x would also depend on the scenarios the problem could be decomposed

The nonanticipative constraints

For each scenario $s = 1, \ldots, S$

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\begin{aligned}
Ax^s &= b \\
T^s x^s + W^s y^s &= h^s
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  whatever the realization of the random event is, the first stage decision should be the same
- The explicit inclusion of the nonanticipativity constraints is at the basis of the decomposition methods specifically tailored for solving two-stage stochastic programming problems.

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- The Value of the Stochastic Solution, **VSS**, measures the possible profit deriving from the solution of the stochastic model with the respect to the deterministic model obtained by replacing the random quantities with their expected values

How to compute the VSS

- Solve the mean-value problem to get a first stage solution $\overline{x}$

$$\begin{aligned} \min \quad & z = c^T x \\ & Ax = b \\ & \overline{T}x = \overline{h} \\ & x \geq 0 \end{aligned}$$

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- Fix the first stage solution at that value $\overline{x}$, and solve for all the scenarios the second stage problem.

$$\begin{aligned} \min \quad & z^s = q^{sT} y^s \\ & T^s \overline{x} + W^s y^s = h^s \\ & y^s \geq 0 \end{aligned}$$

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- VSS measures how much we lose by disregarding uncertainty

The News Vendor Problem

- Let us consider the case of a company that should decide the amount x of a given product to be purchased in order to satisfy some demand d.
- It is a classic problem that should be addressed in the case of perishable or seasonal products (for example, Christmas trees, Flowers on Valentines day...)
- Suppose, for example, that in October we have to decide the number of Christmas trees to buy. Obviously at the moment we do not know what the demand will be. Thus, we have to address a decision problem under uncertainty

The News Vendor Problem

In the scientific literature, such a problem is known as the News Vendor Problem.

The News Vendor Problem: notation

Let us denote by x the number of news paper to buy and assume that at most N news paper can be bought
Let

- c_p be the unitary purchasing price
- c_s be the selling price
- c_f be the price at which the amount in excess to the demand can be sold

The problem aims at determining the number of newspaper to buy so to maximize the profit

The News Vendor Problem

- The demand is evidently unknown, but we assume to know its probability distribution (eventually determined on the basis of the historical data available)
- In order to formulate the problem, we have to determine the relationship between x and $\tilde{d}$
- To this aim we introduce the profit function

$$F(x, \tilde{d}) := \begin{cases} c_s x - c_p x, & \text{if } x \leq \tilde{d} \ , \\ c_s \tilde{d} + c_f (x - \tilde{d}) - c_p x, & \text{if } x > \tilde{d} \ . \end{cases}$$

The function $F(x, \tilde{d})$ can be also written as

$$F(x, \tilde{d}) = c_s \min(x, \tilde{d}) - c_p x + c_f \max(0, x - \tilde{d})$$

The News vendor formulation

Considering the expected value, the problem can be formulated as

$$\begin{aligned} \max \quad & \mathbb{E}[F(x, \tilde{d})] \\ & x \leq N \\ & x \geq 0 \qquad \text{integer} \end{aligned}$$

- If demand distribution is continuous, the objective function is an integral depending on x.
- If we assume that the random demand is discretely distributed, it is possible to determine a solution by solving the problem by an enumeration scheme

The News vendor problem: a toy example

- Let us assume that the demand is discrete and uniformly distributed between 5 and 15 and each outcome has probability $\frac{1}{11}$
- Let $c_p = 20$, $c_s = 25$ and $c_f = 0$ (no salvage value)
- In this case

$$\begin{cases} (c_s - c_p)x, & \text{if } x \leq \tilde{d} \ , \\ c_s\tilde{d} - c_p x, & \text{if } x > \tilde{d} \ . \end{cases}$$

- Hence, the expected profit is:

$$EP(x) = \frac{1}{11}[\sum_{d=5}^{x}(c_s d - c_p x) + \sum_{d=x+1}^{15}(c_s - c_p)x]$$

The News vendor: a toy example

- For instance, if we choose x = 5, profit is 25 in any demand scenario.

$$EP(5) = \frac{1}{11}[(25 * 5 - 20 * 5) * 11] = 25$$

- If we choose x = 6

$$EP(6) = \frac{1}{11}[(25 * 5 - 20 * 6) + (25 * 6 - 20 * 6) * 10] = 27.73$$

The News vendor problem: a toy example

- To choose the best order quantity, we may tabulate the expected profit:

x	EP(x)
5	25.00
6	27.73
7	28.18
8	26.36
9	22.27
10	15.91
11	7.27
12	-3.64
13	-16.82
14	-32.27
15	-50.00

- We see that the optimal solution is not the expected value of demand (10), but a more conservative value (7).

The News vendor: the two-stage formulation

- We assume that the demand follows a discrete distribution
- We denote by d_s the demand under scenario s and by p_s the corresponding probability
- The set of decisions can be divided in two subsets
  1. First-stage decision x – number of news paper to buy
  2. Second-stage decision y^s – number of newspaper sold under scenario s

The News vendor problem: the two-stage formulation

$$
\begin{aligned}
\max \quad & z = -c_p x + \sum_{s=1}^{S} p_s(c_s y^s + c_f(x - y^s)) \\
& x \leq N \\
& y^s \leq x \qquad s = 1, \ldots S \\
& y^s \leq d^s \qquad s = 1, \ldots, S \\
& x \geq 0 \qquad \text{integer} \\
& y^s \geq 0 \qquad s = 1, \ldots S \text{integer}
\end{aligned}
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