Introduction to Stochastic Programming

PhD Course

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- Thus, the decision process can be represented as

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  - $x_t \in \mathbb{R}^{n_t}$ denotes the decisions taken at stage t
  - ω_t represents the uncertainty whose realizations become known at time t

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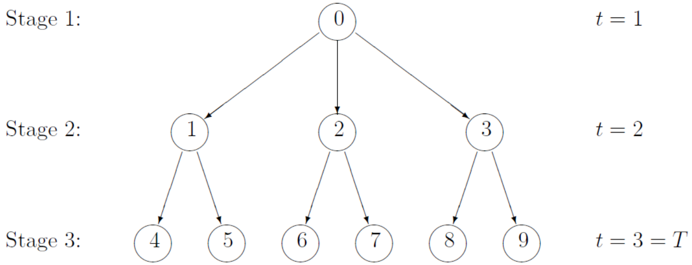
- The multi-stage models can be introduced by considering alternative formulations
- We shall start by introducing the most intuitive one, where the evolution of the uncertain parameters is represented by a scenario tree

Scenario tree for a three-stage problem



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- A **scenario** is path from the root node to a leaf node, i.e. it is a joint realization of the random parameters over all the time stages

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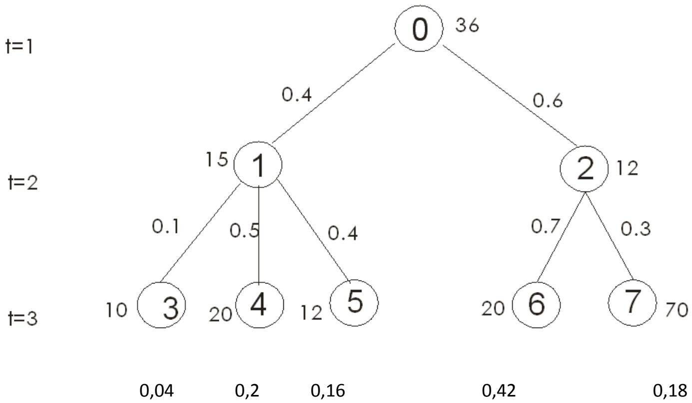
- Let n_1, n_2, n_T be the nodes forming the path from the root node to a leaf node
- The probability associated with the scenario is defined by

$$\rho_{n_1 n_2} * \rho_{n_2 n_3} * \cdots * \rho_{n_{T-1} n_T}$$

The scenario tree: example



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- In practice, we are interested in the decisions that must be implemented here and now, i.e., those corresponding to the root node of the tree
- The other (recourse) decision variables are instrumental to the aim of devising a robust plan, but they are not implemented in practice, as the multistage model is solved on a rolling horizon basis

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- The design of appropriate scenario generation techniques is currently a subject of intensive research since the quality of scenario tree impacts on the recommendations provided by the solution of the stochastic programming formulations
- Given the limited time, we shall not address this important issue in this short course

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- c_n, h^n, T^n, W^n the corresponding matrices and vectors

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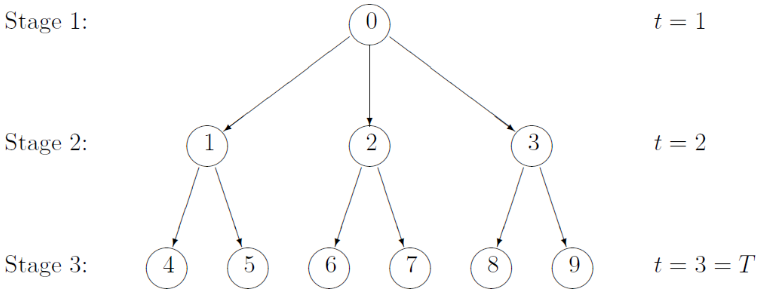
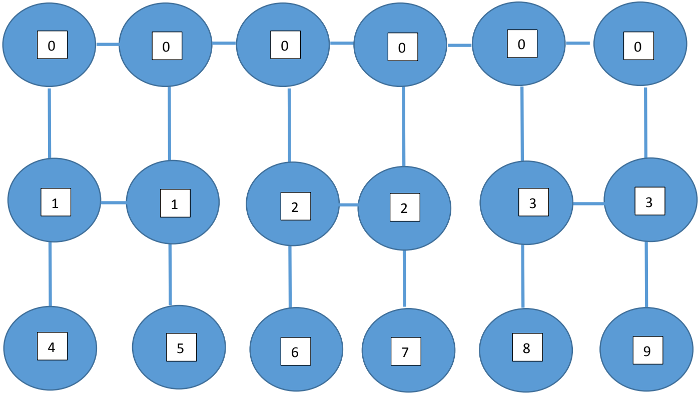
- These constraints state that the decisions associated with two scenarios that share the same history for a given number k of periods, should be identical up to k
- The issue may be understood by looking at the following figure, where horizontal lines correspond to non-anticipativity requirements

The original tree



The split tree



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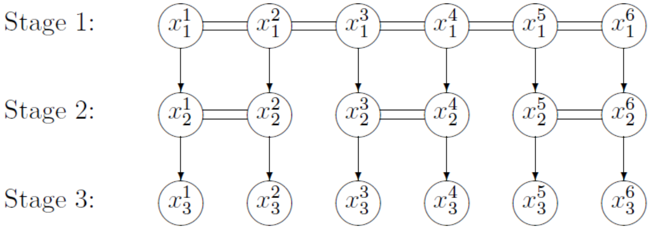

The non-anticipativity condition

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- In our example

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\begin{aligned}
x_1^1 &= x_1^2 = \cdots = x_1^6 = x_0 \\
x_2^1 &= x_2^2 = x_1 \\
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- In general

$$x_{t(n)}^s = x_n \quad \forall n \quad \forall s \in S(n)$$

  - $t(n)$ stage of node n
  - $S(n)$ set of scenarios passing through node n

$$
\begin{aligned}
S(0) &= \{1, 2, 3, 4, 5, 6\} \\
S(1) &= \{1, 2\} \\
S(2) &= \{3, 4\} \\
S(3) &= \{5, 6\}
\end{aligned}
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The split-variable formulation

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\begin{aligned}
\min \quad & z = \sum_{s=1}^{S} p_s \sum_{t=1}^{T} c_t^s x_t^s \\
& A x_1^s = b \quad s = 1, \dots S \\
& T_t^s x_{t-1}^s + W_t^s x_t^s = h_t^s \;\; \forall t \geq 2, s = 1, \dots S \\
& x_n - x_{t(n)}^s = 0 \quad \forall n \in \mathcal{N} \quad \forall s \in S(n) \\
& x_t^s \geq 0 \quad t = 1, \dots T \quad s = 1, \dots, S
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- The main motivation is related to high volatility of financial markets that makes the explicit inclusion of the uncertainty in the mathematical model mandatory to derive financial plans that can be used in practice
- In particular, the multistage framework offers a valuable paradigm allowing to define optimal investment plans that can revised over the time as new information becomes available

A multi-stage model for Portfolio Optimization

- We consider a planning horizon divided into a number of elementary periods $t = 1, 2, \dots T$
- At each period t of the planning horizon, the investor must decide:
  - The amount of security i to be purchased B_{it}
  - The amount of security i to sell S_{it}
  - The amount of security i to be maintained in the portfolio H_{it}
  - The monetary amount to invest in a risk-free asset v_t

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$$(1-g)\sum_{i=1}^{N} P_{it}S_{it} + F_t + (1+r_t)v_{t-1} =$$
$$(1+g)\sum_{i=1}^{N} P_{it}B_{it} + L_t + v_t \qquad t = 2, \ldots, T-1$$

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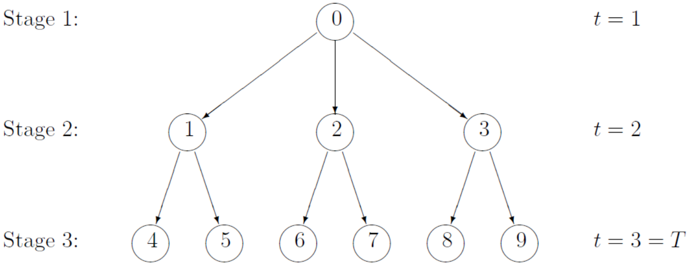
The deterministic model

- **The objective function**

$$max \quad W_T = (1-g)\sum_{i=1}^{N} P_{iT} H_{iT-1} + (1+r_T) v_{T-1} + F_T - L_T$$

Scenario tree for a three-stage problem



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  - P_{in} the price of asset i
  - L_n the liability
  - F_n the available fund to invest
  - r_n the risk-free interest rate

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For each node $n \in \mathcal{N}$ we denote by

- S_{in} the amount of security i to sell
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$$H_{i0} = InitHold_i + B_{i0} - S_{i0} \quad i = 1, \dots N$$
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$$(1-g)\sum_{i=1}^{N} P_{i0}S_{i0} + F_0 = (1+g)\sum_{i=1}^{N} P_{i0}B_{i0} + L_0 + v_0$$
$$(1-g)\sum_{i=1}^{N} P_{in}S_{in} + F_n + (1+r_n)v_{a(n)} =$$
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$$W_n = (1-g)\sum_{i=1}^{N} P_{in} H_{ia(n)} + (1+r_n) v_{a(n)} + F_n - L_n \ \forall n \text{ leaf node}$$

- **The objective function**

$$\max z = \sum_{n \ \in \{\text{leaf nodes}\}} p_n * W_n$$

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- **Monetary balance constraints**

$$(1-g)\sum_{i=1}^{N} P_{i1}^s S_{i1}^s + F_1^s = (1+g)\sum_{i=1}^{N} P_{i1}^s B_{i1}^s + L_1^s + v_1^s \ \forall s$$

$$(1-g)\sum_{i=1}^{N} P_{it}^s S_{it}^s + F_t^s + (1+r_t^s)v_{t-1}^s =$$
$$(1+g)\sum_{i=1}^{N} P_{it}^s B_{it}^s + L_t^s + v_t^s \quad t = 2, \dots T-1 \ \forall s$$

The multi-stage node formulation

The multi-stage node formulation

- **Definition of the last stage**

$$W_T^s = (1-g)\sum_{i=1}^{N} P_{iT}^s H_{iT-1}^s + (1+r_T^s)v_{T-1}^s + F_T^s - L_T^s \ \forall s$$

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- **The non-anticipativity constraints**

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- **The non-anticipativity constraints**

$$\begin{aligned}
H_{in} &= H_{it(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)\\
S_{in} &= S_{it(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)\\
B_{in} &= B_{it(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)\\
v_n &= v_{t(n)}^s \quad \forall n \in \mathcal{N} \quad s \in S(n)
\end{aligned}$$

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\end{aligned}$$

- **The objective function**

$$\max z = \sum_{s=1}^{S} p_s W_T^s$$

Exercise

- Let us consider a classical production problem defined over a time horizon divided in a given number T of periods.
- For the sake of simplicity, we shall consider a single product.
- At each period t, we have to decide the amount to produce so to satisfy a given demand
- We consider the possibility to stock the eventual amount in excess
- Define the deterministic and the stochastic models assuming that the evolution of the uncertain demand can be represented as the scenario tree reported below

The deterministic version

For each time period $t = 1, \ldots, T$ we denote by

- c_t the unitary production cost
- π_t the unitary inventory cost
- d_t the demand
- x_t the amount to produce
- I_t the amount to stock

$$\min z = \sum_{t=1}^{T} (c_t * x_t + \pi_t * I_t)$$

$$I_t = I_{t-1} + x_t - d_t \qquad t = 1, \ldots T$$

$$x_t \geq 0, \qquad I_t \geq 0 \qquad t = 1, \ldots, T$$

The scenario tree: example