Introduction to Stochastic Programming

PhD Course

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Recourse Philosophy

In the general recourse models, we guarantee feasibility by the correction action of the second-stage variables

$$
\begin{aligned}
\min \quad & c^T x + \mathbb{E}_\omega[q^T(\omega)y(\omega)] \\
& Ax = b \\
& T(\omega)x + W(\omega)y(\omega) = h(\omega) \quad \forall\, \omega \in \Omega \\
& y(\omega) \geq 0 \quad \forall\, \omega \in \Omega \\
& x \geq 0
\end{aligned}
$$

Probabilistic Constraints

We admit the violation of the stochastic constraints provided that this occurs with a very low level of probability
Thus, the stochastic constraints

$$T(\omega)x \geq h(\omega)$$

are replaced by

$$\mathbb{P}(T(\omega)x \geq h(\omega)) \geq \alpha$$

We impose the satisfaction of the stochastic constraints with a level of probability α typically high
In other terms, we admit its violation with probability $(1 - \alpha)$

Probabilistic Constraints: formulation

Assuming that the cost coefficient are deterministic (or replacing the random variables with their expected values), we get

$$
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\min \quad & c^T x \\
& \mathbb{P}(T(\omega)x \geq h(\omega)) \geq \alpha \\
& Ax = b \\
& x \geq 0
\end{aligned}
$$

- The choice of the α value is up to the decision maker
- The higher the value the worse the objective function value

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  Define the optimal portfolio composition so to guarantee that a given target value is met with high confidence

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- Localization of the emergency systems
  Determine the location of the emergency stations so to satisfy the random service demand with a high reliability level
- Portfolio optimization
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- Routing problems
  Determine the set of routes used by a fleet of vehicles to serve a given set of customers with random demand with high probability level
- . . .

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- **individually imposed**

$$\mathbb{P}(T_i(\omega)x \geq h_i(\omega)) \geq \alpha_i \qquad i = 1, \ldots, m_2$$

where $T_i(\omega)$ denotes the i-th row of the stochastic matrix and $h_i(\omega)$ is the i-th component of the vector $h(\omega)$

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The choice of joint or individual chance constraints depends on the specific application

The special case of normal distribution

- The probabilistic constraints are very difficult to deal with because the feasible set is typically non convex
- We derive a deterministic equivalent formulation in the specific case of a single probabilistic constraint
- We shall assume that the components $t_j(\omega)$ of the matrix T follow a Normal distribution

$$t_j(\omega) \sim N(\mu_j, \sigma_j^2)$$

- For the sake of simplicity, we shall assume that h is a deterministic value

The deterministic equivalent reformulation

In order to derive the deterministic equivalent reformulation, we use the properties of the normal random variables
We recall that

$$\sum_{j=1}^{n} t_j(\omega) x_j$$

is a normal random variable with expected value equal to

$$\sum_{j=1}^{n} \mu_j x_j$$

and variance

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

where σ_{ij} is the covariance between t_i and t_j

The deterministic equivalent reformulation

We now "‘normalize"’ the constraint

$$P\left(\underbrace{\frac{\sum_{j=1}^n t_j(\omega)x_j - \sum_{j=1}^n \mu_j x_j}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}x_i x_j}}}_{\tilde{z}} \geq \frac{h - \sum_{j=1}^n \mu_j x_j}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}x_i x_j}}\right) \geq \alpha$$

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$$P\left(\tilde{z} \geq \frac{h - \sum_{j=1}^n \mu_j x_j}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j}}\right) \geq \alpha$$

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$$1 - P\left(\tilde{z} \leq \frac{h - \sum_{j=1}^{n} \mu_j x_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j}}\right) \geq \alpha$$

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$$\Phi_{\tilde{z}}\left(\frac{h - \sum_{j=1}^{n} \mu_j x_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j}}\right) \leq (1 - \alpha)$$

$$\frac{h - \sum_{j=1}^{n} \mu_j x_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \leq \Phi^{-1}(1 - \alpha)$$

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$$\sum_{j=1}^{n} \mu_j x_j \geq h - \Phi^{-1}(1-\alpha)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}$$

Here $\Phi^{-1}(1-\alpha)$ represents the $(1-\alpha)$ quantile of the normal standard distribution function

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The corresponding model belongs to the class of nonlinear integer mathematical programming problems

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- The problem can be modeled as

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\begin{aligned}
\max \quad & \sum_{i=1}^{n} E[R_i]x_i \\
& P(\sum_{i=1}^{n} R_i x_i \geq \eta) \geq \alpha \\
& \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0 \quad \forall i
\end{aligned}
$$

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$$\bigcap_{s \in I} \Gamma^s$$

$$\{I \subseteq \{1, \dots, S\} | \sum_{s \in I} p_s \geq \alpha\}$$

Reformulation in the case of discrete distributions

- Disjunctive reformulation

$$\bigcup_{I \in \Delta} \bigcap_{s \in I} \Gamma^s$$

where

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- We introduce a binary variable z^s for each scenario s

$$z^s = \begin{cases} 0, & \text{if } \Gamma^s \text{ is satisfied} \\ 1, & \text{otherwise.} \end{cases}$$

Reformulation in the case of discrete distributions

$$
\begin{aligned}
\min \quad & c^T x \\
& T^s x + M^s z^s \geq h^s \quad s = 1, \ldots, S \\
& \sum_{s=1}^{S} p_s * z^s \leq (1-\alpha) \\
& x \geq 0 \\
& z^s \in \{0, 1\}
\end{aligned}
$$

Here M^s denotes a big positive number

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- Some references

Some references

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