Online Appendix for the Paper

Stable Model Semantics of Abstract Dialectical Frameworks Revisited: A Logic Programming Perspective

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A Proofs

Theorem 1. *Let Π be a DLPBF, S be the set of stable models of Π according to Son and Pontelli (2007), and S' be the set of P-stable models of Π, then $S \equiv_{At(\Pi)} S'$.*

Proof. Let Π be a DLPBF. The proof is structured in two parts providing mappings from the two sets of stable models.

Part I. Let I be a stable model of Π according to Son and Pontelli (2007). We shall show that I is a P-stable model of Π. Let $K_0 = \emptyset$, $K_{i+1} = K^I_\Pi(K_i)$, $T_0 = \emptyset$, $T_{i+1} = T^I_{\Pi'}(T_i)$), where $i \geq 0$ and $\Pi' = P(\mathit{\Pi}, I)$. We use induction to show that $K_i = T_i \cap At(\Pi)$ holds for all $i \geq 0$. The base case is trivial. Let us assume that the claim hold for some $i \geq 0$. For each $p \in K_{i+1} = K^I_\Pi(K_i)$ there is $p \leftarrow f \in \Pi$ such that $[f]_I = \mathbf{T}$ and $[f]_{(K_i,I)} = \mathbf{T}$. It follows that $[\mathit{keep\_all}(f, I)]_{(T_i,I)} = \mathbf{T}$, and therefore $p \in T^I_{\Pi'}(T_i) = T_{i+1}$. For each $p \in At(\Pi) \setminus K_{i+1} = At(\Pi) \setminus K^I_\Pi(K_i)$ we have that, for all $p \leftarrow f \in \Pi$, $[f]_{(K_i,I)} = \mathbf{F}$. Hence, $[\mathit{keep\_all}(f, I)]_{(T_i,I)} = \mathbf{F}$, and therefore $p \notin T^I_{\Pi'}(T_i) = T_{i+1}$.

Part II. Consider now a P-stable model I of Π. We shall show that I is a stable model of Π according to Son and Pontelli (2007). Let $T_0 = \emptyset$, $T_{i+1} = T^I_{\Pi'}(T_i)$, $K_0 = \emptyset$, $K_{i+1} = K^I_\Pi(K_i)$, where $i \geq 0$ and $\Pi' = P(\mathit{\Pi}, I)$. We use induction to show that $T_i \cap At(\Pi) = K_i$ holds for all $i \geq 0$. The base case is trivial. Let us assume that the claim hold for some $i \geq 0$. For each $p \in T_{i+1} \cap At(\Pi) = T^I_{\Pi'}(T_i) \cap \mathit{\Pi}$ we have $[\mathit{keep\_all}(\Pi_p, I)]_{(T_i,I)} = \mathbf{T}$, and therefore $[\Pi_p]_{(K_i,I)} = \mathbf{T}$. We conclude that $p \in K_{i+1}$. For each $p \in At(\Pi) \setminus T_{i+1}$ we have $[\mathit{keep\_all}(\Pi_p, I)]_{(T_i,I)} \neq \mathbf{T}$, which implies that $[\Pi_p]_{(K_i,I)} \neq \mathbf{T}$. Hence, we conclude $p \notin K_{i+1}$. □

Theorem 2. *Let Π be a DLPBF, S be the set of stable models of Π according to Gelfond and Zhang (2014), and S' be the set of G-stable models of Π, then $S \equiv_{At(\Pi)} S'$.*

Proof. The reduct in equation (5) is obtained by removing rules whose Boolean functions assign $\mathbf{F}$ to I, and by replacing each remaining Boolean function with the intersection of its domain and I. If I is the least model of the reduct then I is a stable model. Similarly, equation (9) replaces Boolean functions assigning $\mathbf{F}$ to I by a Boolean function whose image is $\{\mathbf{F}\}$, and each remaining Boolean function by a Boolean function that is $\mathbf{F}$ everywhere but for the intersection of its domain and I. □

Theorem 3. *Let $F = \langle \mathcal{S}, E, C\rangle$ be an ADF, S be the set of stable models of F according to Brewka* et al. *(2013), and S' be the set of B-stable models of $\mathit{def}(F)$, then $S \equiv_\mathcal{S} S'$.*

Proof. First of all, note that the reduced ADF F' in Definition 2 coincides with $\Pi' = B(\Pi, I)$ in Definition 6:

- in F' the set of propositions is restricted to I, while in Π' the image of Π'_p is $\{\mathbf{F}\}$ for all $p \in \mathcal{V} \setminus I$;
- $(\mathit{par}(s) \setminus A) \cap I = \emptyset$ is equivalent to $A \cap \mathit{par}(s) \subseteq I$, i.e., $\mathit{dom}(\Pi_s) \cap A \subseteq \mathit{dom}(\Pi_s) \cap I$ (see equation 8), for all $s \in \mathcal{S}$ and $A \subseteq \mathit{par}(s)$.

We have to show that $T^I_{\Pi'} \Uparrow \emptyset = \Gamma_{F'} \Uparrow (\emptyset, \mathcal{S})$ whenever $I = T^I_{\Pi'} \Uparrow \emptyset$ or $I = \Gamma_{F'} \Uparrow (\emptyset, \mathcal{S})$.

Let $\emptyset = X_0 \subseteq \cdots \subseteq X_m = X_{m+1}$ $(m \geq 0)$ be such that $X_i = T^I_{\Pi'}(X_{i-1})$ for $i \in [1..m]$. Let $(L_0, U_0), \ldots, (L_{n+1}, U_{n+1})$ be such that $(L_0, U_0) = (\emptyset, \mathcal{A})$, $L_{i-1} \subseteq L_i$, $U_{i-1} \supseteq U_i$, $(L_i, U_i) = \Gamma_{\Pi'}(L_{i-1}, U_{i-1})$, and $(L_n, U_n) = (L_{n+1}, U_{n+1})$ (for $i \in [1..n]$). We observe that $U_i \subseteq I$ for $i \in [1..n]$ because $[\Pi'_p]_{(L_0,U_0)} = \mathbf{F}$ for each $p \notin I$ by construction.

We now use induction to show $X_i \subseteq L_n$ and $X_i \subseteq U_n$ $(i \geq 0)$. The base case is vacuously true. For $p \in X_{i+1} \setminus X_i$ $(i \geq 1)$ we have $[\Pi'_p]_{(X_i,I)} = \mathbf{T}$; since $X_i \subseteq L_n$ and $X_i \subseteq U_n$ by the induction hypothesis, and $U_n \subseteq I$, we have $[\Pi'_p]_{(L_n,U_n)} = \mathbf{T}$. Hence, $I = X_m$ immediately implies $(L_n, U_n) = I$. On the other hand, if $I = (L_n, U_n)$ then we can use induction to show $L_i \subseteq X_m$ $(i \geq 0)$. The base case is vacuously true. For $p \in L_{i+1} \setminus L_i$ we have $[\Pi'_p]_{(L_i,U_i)} = \mathbf{T}$, which implies $[\Pi'_p]_{(X_m,I)} = \mathbf{T}$, i.e., $p \in X_m$. □

Theorem 4. *Let $F = \langle \mathcal{S}, E, C\rangle$ be an ADF, S be the set of stable models of F according to Strass (2013), and S' be the set of S-stable models of $\mathit{def}(F)$, then $S \equiv_\mathcal{S} S'$.*

Proof. Replacing F by $\Pi = \mathit{def}(F)$ in (3), we have:

$$G^I_\Pi(X) := \{s \in \mathcal{V} \mid \exists A \subseteq X,\ \Pi_s(A) = \mathbf{T},\ (\mathit{dom}(\Pi_s) \setminus A) \cap I = \emptyset,\ (\mathit{dom}(\Pi_s) \setminus A) \cap X = \emptyset\}.$$

We first observe that $(dom(\Pi_s) \setminus A) \cap I = \emptyset$ is equivalent to $dom(\Pi_s) \cap I \subseteq dom(\Pi_s) \cap A$. Similarly, $(dom(\Pi_s) \setminus A) \cap X = \emptyset$ is equivalent to $dom(\Pi_s) \cap X \subseteq dom(\Pi_s) \cap A$, and combining with $A \subseteq X$ we have $dom(\Pi_s) \cap A = dom(\Pi_s) \cap X$, i.e., $A = X$. Hence, we have:

$$\begin{aligned} G^I_\Pi(X) = \{s \in \mathcal{V} \mid \ & \exists A \subseteq X,\ \Pi_s(A) = \mathbf{T}, \\ & dom(\Pi_s) \cap A \supseteq dom(\Pi_s) \cap I, \\ & dom(\Pi_s) \cap A = dom(\Pi_s) \cap X\} \\ = \{s \in \mathcal{V} \mid \ & \Pi_s(X) = \mathbf{T}, \\ & dom(\Pi_s) \cap X \supseteq dom(\Pi_s) \cap I\}. \end{aligned}$$

To complete the proof we just note that for $X \subseteq I$ condition $dom(\Pi_s) \cap X \supseteq dom(\Pi_s) \cap I$ is equivalent to $dom(\Pi_s) \cap X = dom(\Pi_s) \cap I$, and therefore $\Pi_s(X) = \Pi_s(I)$. □

Theorem 5. *A total interpretation I is a B-stable model of a DLPBF Π if and only if I is a P-stable model of Π.*

Proof. We shall show that $T^I_{B(\Pi,I)}(X) = T^I_{P(\Pi,I)}(X)$, for all $X \subseteq I \subseteq \mathcal{V}$.

($\subseteq$) Let $p \in T^I_{B(\Pi,I)}(X)$, and $p \leftarrow f \in \Pi$. Hence, $[f^{I,\subseteq}]_{(X,I)} = \mathbf{T}$, which in turn implies $[f^I]_{(X,I)} = \mathbf{T}$, and therefore $p \in T^I_{P(\Pi,I)}(X)$.

($\supseteq$) Let $p \in T^I_{P(\Pi,I)}(X)$, and $p \leftarrow f \in \Pi$. Hence, $[f^I]_{(X,I)} = \mathbf{T}$. Since $X \subseteq I$, $dom(f) \cap X \subseteq dom(f) \cap I$. We can conclude $[f^{I,\subseteq}]_{(X,I)} = \mathbf{T}$, and therefore $p \in T^I_{B(\Pi,I)}(X)$. □

Theorem 6. *A total interpretation I is an S-stable model of a DLPBF Π if and only if I is a G-stable model of Π.*

Proof. We shall show that $T^I_{S(\Pi,I)}(X) = T^I_{G(\Pi,I)}(X)$, for all $X \subseteq I \subseteq \mathcal{V}$.

($\subseteq$) Let $p \in T^I_{S(\Pi,I)}(X)$, and $p \leftarrow f \in \Pi$. Hence, $[f^{I,\supseteq}]_{(X,I)} = \mathbf{T}$. Since $X \subseteq I$, $dom(f) \cap X \subseteq dom(f) \cap I$, and combining with $[f^{I,\supseteq}]_{(X,I)} = \mathbf{T}$ we have $dom(f) \cap X = dom(f) \cap I$. We can conclude $[f^{I,=}]_{(X,I)} = \mathbf{T}$, and therefore $p \in T^I_{G(\Pi,I)}(X)$.

($\supseteq$) Let $p \in T^I_{G(\Pi,I)}(X)$, and $p \leftarrow f \in \Pi$. Hence, $[f^{I^=}]_{(X,I)} = \mathbf{T}$, which in turn implies $[f^{I,\supseteq}]_{(X,I)} = \mathbf{T}$, and therefore $p \in T^I_{S(\Pi,I)}(X)$. □

Theorem 7. *If a total interpretation I is a G-stable model of a DLPBF Π then I is a P-stable model of Π.*

Proof. Let $I = T^I_{G(\Pi,I)} \Uparrow \emptyset$. Let $\emptyset = X_0 \subseteq \cdots \subseteq X_m = X_{m+1}$ ($m \geq 0$) be such that $X_i = T^I_{P(\Pi,I)}(X_{i-1})$ for $i \in [1..m]$. Let $\emptyset = Y_0 \subseteq \cdots \subseteq Y_n = I$ ($n \geq 0$) be such that $Y_i = T^I_{G(\Pi,I)}(Y_{i-1})$ for $i \in [1..n]$. We use induction to show that $X_i \subseteq I$ ($i \geq 0$). The base case is vacuously true. For $p \in X_{i+1} \setminus X_i$ we have $[\Pi_p]_{(X_i,I)} = \mathbf{T}$, which implies $[G(\Pi,I)_p]_I = \mathbf{T}$, and thus $p \in I$. We now use induction to show that $Y_i \subset X_m$. The base case is vacuously true. For $p \in Y_{i+1} \setminus Y_i$ we have $[G(\Pi,I)_p]_{(Y_i,I)} = \mathbf{T}$, which combined with $X_m \subseteq I$ and with the induction hypothesis $Y_i \subseteq X_m$ gives $[\Pi_p]_{(X_m,I)} = \mathbf{T}$. Therefore, $p \in X_m$, and we proved $X_m = I$. □

Corollary 1. *If a total interpretation I is an S-stable model of a DLPBF Π then I is a B-stable model of Π.*

Proof. It is a consequence of Theorems 5, 6, and 7. □

Theorem 8. *Let Π be a DLPBF, S be the set of S- or G-stable models of Π, and S' be the set of B- or P-stable models of $s2b(\Pi)$, then $S \equiv_{At(\Pi)} S'$.*

Proof. Let Π be a DLPBF. The proof is structured in two parts providing mappings from the two sets of stable models.

Part I. Let I be a G-stable model of Π. Define $I' = I \cup \{p^F \mid p \in At(\Pi) \setminus I\} \cup \{p^= \mid p \in At(\Pi)\}$. We shall show that I' is a P-stable model of $s2b(\Pi)$. Let $G_0 = \emptyset$, $G_{i+1} = T^I_{G(\Pi,I)}(G_i)$, $P_0 = \emptyset$, $P_{i+1} = T^I_{P(s2b(\Pi),I)}(P_i)$, for all $i \geq 0$. Note that $p \in P_1$ if and only if $p \in I$, and $p^F \in P_1$ if and only if $p \in At(\Pi) \setminus I$. We have that $G_i = \{p \in I \mid p^= \in P_{i+1}\}$ holds for all $i \geq 0$.

Part II. Let I be a P-stable model of $s2b(\Pi)$. We shall show that $I \cap At(\Pi)$ is a G-stable model of Π. In fact, note that $(I \cap At(\Pi)) \cup \{p^F \mid p \in At(\Pi) \setminus I\} \cup \{p^= \mid p \in At(\Pi)\} = I$ because of the following reasons:

- Atoms p, p^F are used to guess an interpretation $I \cap At(\Pi)$ for Π.
- Atom $\bot$ must be false in all P-stable models of $s2b(\Pi)$, and therefore $I \cap At(\Pi)$ must be a model of Π.
- The falsity of $\bot$ also implies that $p^= \in I$ for all $p \in At(\Pi)$.

Hence, as observed in Part I, we can conclude that $T^I_{P(s2b(\Pi),I)} \Uparrow \emptyset = T^I_{G(\Pi,I)} \Uparrow \emptyset$ holds. □

Theorem 9. *Let Π be an LPBF, S be the set of stable models of Π according to Son and Pontelli (2007), and S' be the set of P-stable models of $def(\Pi)$, then $S \equiv_{At(\Pi)} S'$.*

Proof. Let Π be an LPBF. The proof is structured in two parts providing mappings from the two sets of stable models.

Part I. Let I be a stable model of Π according to Son and Pontelli (2007). Define $I' = I \cup \{aux_f \mid p \leftarrow f \in \Pi,\ [f]_I = \mathbf{T}\}$. We shall show that I' is a P-stable model of $def(\Pi)$. Let $K_0 = \emptyset$, $K_{i+1} = K^I_\Pi(K_i)$, $T_0 = \emptyset$, $T_{i+1} = T^{I'}_{\Pi'}(T^{I'}_{\Pi'}(T_i))$, where $i \geq 0$ and $\Pi' = P(def(\mathit{\Pi}), I')$. We use induction to show that $K_i = T_i \cap At(\Pi)$ holds for all $i \geq 0$. The base case is trivial. Let us assume that the claim hold for some $i \geq 0$. For each $p \in K_{i+1} = K^I_\Pi(K_i)$ there is $p \leftarrow f \in \Pi$ such that $[f]_I = \mathbf{T}$ and $[f]_{(K_i,I)} = \mathbf{T}$. It follows that $aux_f \in I'$, $aux_f \leftarrow f \in \Pi'$, $aux_f \in T^{I'}_{\Pi'}(T_i)$, and therefore $p \in T^{I'}_{\Pi'}(T^{I'}_{\Pi'}(T_i)) = T_{i+1}$. For each $p \in At(\Pi) \setminus K_{i+1} = At(\Pi) \setminus K^I_\Pi(K_i)$ we have that, for all $p \leftarrow f \in \Pi$, $[f]_{(K_i,I)} = \mathbf{F}$. Hence, $aux_f \notin T^{I'}_{\Pi'}(T_i)$, and therefore $p \notin T^{I'}_{\Pi'}(T^{I'}_{\Pi'}(T_i)) = T_{i+1}$.

Part II. Consider now a P-stable model I of $def(\Pi)$. We shall show that $I' = I \cap At(\Pi)$ is a stable model of Π according to Son and Pontelli (2007). Let $T_0 = \emptyset$, $T_{i+1} = T^I_{\Pi'}(T^I_{\Pi'}(T_i))$, $K_0 = \emptyset$, $K_{i+1} = K^{I'}_{\Pi}(K_i)$, where $i \geq 0$ and $\Pi' = P(def(\Pi), I)$. We use induction to show that $T_i \cap At(\Pi) = K_i$ holds for all $i \geq 0$. The base case is trivial. Let us assume that the claim hold for some $i \geq 0$. For each $p \in T_{i+1} \cap At(\Pi) = T^I_{\Pi'}(T^I_{\Pi'}(T_i)) \cap \Pi$ we have $[\Pi'_p]_{(T_i,I)} = \mathbf{T}$, and therefore there is $aux_f \in T^I_{\Pi'}(T_i)$ such that $p \leftarrow f \in \Pi$. We conclude that $[f]_{(T_i,I)} = \mathbf{T}$, and hence $p \in K_{i+1}$. For each $p \in At(\Pi) \setminus T_{i+1}$ we have $[\Pi'_p]_{(T_i,I)} \neq \mathbf{T}$, which implies that no aux_f such that $p \leftarrow f \in \Pi$ belongs to $T^I_{\Pi'}(T_i)$. Hence, we conclude $[f]_{(T_i,I)} \neq \mathbf{T}$, and therefore $p \notin K_{i+1}$. □

Theorem 10. *Let Π be an LPBF, S be the set of stable models of Π according to Gelfond and Zhang (2014), and S' be the set of B- or P-stable models of $s2b(\Pi)$, then $S \equiv_{At(\Pi)} S'$.*

Proof. The proof is essentially as that for Theorem 8. We have just to note that (14) can be replaced by

$$G(\Pi, I) = \{p \leftarrow keep\_equal(f, I) \mid p \leftarrow f \in \Pi\}$$

to restate G-stable models for LPBFs. □

Theorem 11. *Let Π be a DLPBF, S be the set of B-stable models of Π, and S' be the set of stable models of $adf(\Pi)$ according to Brewka* et al.*, then $S \equiv_{\mathcal{S}} S'$.*

Proof. Note that $def(adf(\Pi)) = \Pi$. Hence, the claim follows by Theorem 3. □

Theorem 12. *Let Π be a DLPBF, S be the set of S-stable models of Π, and S' be the set of stable models of $adf(\Pi)$ according to Strass, then $S \equiv_{\mathcal{S}} S'$.*

Proof. Note that $def(adf(\Pi)) = \Pi$. Hence, the claim follows by Theorem 4. □

Corollary 2. *Let Π be an LPBF, S be the set of stable models of Π according to Brewka* et al.*, and S' be the set of stable models of $adf(def(\Pi))$ according to Brewka* et al.*, then $S \equiv_{At(\Pi)} S'$.*

Proof. The claim follows from Theorems 9 and 11. □

Corollary 3. *Let Π be an LPBF, S be the set of stable models of Π according to Strass, and S' be the set of stable models of $adf(s2b(\Pi))$ according to Brewka* et al.*, then $S \equiv_{At(\Pi)} S'$.*

Proof. The claim follows from Theorems 10 and 12. □

Corollary 4. *Let Π be a DLPBF, and I be a total interpretation. If I is a G-, S-, P- or B-stable model then I is an F-stable model.*

Proof. For programs with atomic heads, as those considered in this paper, Theorem 5 in [Shen and Wang, 2012] claims equivalence of their stable models with stable models in Definition 3 of [Son *et al.*, 2007], which is in turn equivalent to stable models in [Son and Pontelli, 2007]. That each stable model by Son and Pontelli (2007) is an F-stable model is stated in Theorem 2 in [Son and Pontelli, 2007]. □