Online Appendix for the Paper

A MaxSAT Algorithm Using Cardinality Constraints of Bounded Size

published in IJCAI-15

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A Proof of Correctness

Lemma 1. *Let ϕ be a propositional theory, w be a cost function, and $\{x_0, \ldots, x_n\} \subseteq \mathcal{V}$ be such that $n \geq 0$ and $w(x_i) \geq 1$ for all $i \in [0..n]$. Let $m = \min_{i \in [0..n]} w(x_i)$, and $r_0, \ldots, r_n$ be fresh variables. Define ϕ' and w' as follows:*

$$\begin{aligned}\phi' := \phi \cup &\{x_0 + \cdots + x_n + \neg r_0 + \cdots + \neg r_n \geq n+1\} \\ \cup &\{r_i \rightarrow r_{i+1} \mid i \in [0..n-1]\}\end{aligned} \tag{4}$$

$$w'(x) := \begin{cases} w(x) - m & \text{if } x \in \{x_0, \ldots, x_n\} \\ m & \text{if } x \in \{r_0, \ldots, r_n\} \\ w(x) & \text{otherwise.} \end{cases} \tag{5}$$

The following relation holds: $\mathcal{O}(\phi, w) \equiv_{vars(\phi)} \mathcal{O}(\phi', w')$.

Proof. For each interpretation I of ϕ define $I' := I \cup \{r_i \mid i \in [0..n],\ I \models x_0 + \cdots + x_n \geq n + 1 - i\}$. The proof is structured as follows:

i. $I \models \phi$ if and only if $I' \models \phi'$;
ii. for each $J \in \mathcal{O}(\phi', w')$ there is some I such that $J = I'$;
iii. $w(I) = w'(I')$.

(Proof of i) It is enough to prove that $I' \models \phi' \setminus \phi$. In fact, if $t = |I \cap \{x_0, \ldots, x_n\}|$, then $\{\neg r_i \mid r_i \notin I'\} = \{\neg r_0, \ldots, \neg r_{n-t}\}$ by construction of I', i.e., $|\{\neg r_i \mid r_i \notin I'\}| = n + 1 - t$. Hence, $I' \models x_0 + \cdots + x_n + \neg r_0 + \cdots + \neg r_n \geq n + 1$. Moreover, note that $I \cap \{r_0, \ldots, r_n\} = \{r_{n+1-t}, \ldots, r_n\}$, and therefore $I' \models r_i \rightarrow r_{i+1}$, for all $i \in [0..n-1]$.

(Proof of ii) Let us prove the contrapositive: If there is no I such that $J = I'$ then $J \notin \mathcal{O}(\phi', w')$. Suppose also that $J \models \phi'$. Hence, set $R = \{r_i \mid i \in [0..n],\ r_i \notin J,\ J \cap vars(\phi) \models x_0 + \cdots + x_n \geq n+1-i\}$ is nonempty. Moreover, $J \setminus R \models \phi'$, and $w'(J \setminus R) = w'(J) - m \cdot |R|$ (recall that $m = w(r_i)$, for all $i \in [0..n]$). Since $m \geq 1$ by assumption, we have $w'(J \setminus R) < w'(J)$, and we are done.

(Proof of iii) Without loss of generality, let us assume that there is a $j \in [0..n+1]$ such that $I \cap \{x_0, \ldots, x_n\} = \{x_0, \ldots, x_{j-1}\}$. Recall, again, that $m = w(r_i)$, for all $i \in [0..n]$. We have $w(I) = w'(I) + (n - j + 1) \cdot m$, where $(n - j + 1) \cdot m$ is the cost subtracted from w when defining w'. Now note that $I' \cap \{r_0, \ldots, r_n\} = \{r_0, \ldots, r_{j-1}\}$, and therefore $w(I) = w'(I) + w'(\{r_j, \ldots, r_n\}) = w'(I')$. □

Theorem 1. *Algorithm 1 is correct, that is, for any consistent formula ϕ and cost function w, it outputs a pair $(I, w(I))$, where $I \in \mathcal{O}(\phi, w)$. Correctness holds also if function k-ProcessCore is used, for all $k \geq 1$.*

Proof. We prove that at each step optimum stable models are preserved. Let ϕ be the theory at the current step, and $C = \{x_0, \ldots, x_n\}$, for some $n \geq 0$, be the unsatisfiable core returned by function Solve. Let ϕ' be as in Lemma 1. Since $\{x_0, \ldots, x_n\}$ are in an unsatisfiable core of ϕ, we have that any model I of ϕ' is such that $r_0 \notin I$. Hence, constraint (4) can be replaced by the constraint in function ProcessCore, and clause $r_0 \rightarrow r_1$ can be removed. The resulting theory is exactly ϕ modified by function ProcessCore. From Lemma 1 we have $\mathcal{O}(\phi, w) = \mathcal{O}(\phi', w')$. Moreover, as already observed, $I \in \mathcal{O}(\phi, w)$ implies $I \not\models r_0$, i.e., $w(I) \geq m$, where $m = \min_{i \in [0..n]} w(x_i)$. And indeed variable $lower\_bound$ is increased by m in Algorithm 1.

As for k-ProcessCore, let $C = \{x_0, \ldots, x_{n \cdot k}\}$ $(n \geq 0)$, and $m = \min_{x \in C} w(x)$. We shall show the claim by n applications of Lemma 1. Let ϕ_1, w_1 be ϕ', w' in Lemma 1 applied on ϕ, w and $\{x_0, \ldots, x_k\}$, where variable r_0 is renamed c_1. For all $i \geq 1$, let ϕ_{i+1}, w_{i+1} be ϕ', w' in Lemma 1 applied on ϕ_i, w_i and $\{c_i, x_{i \cdot k+1}, \ldots, x_{(i+1)k}\}$, where variable r_0 is renamed c_{i+1}, and variables $r_1, \ldots, r_n$ are renamed $r_{i \cdot k+1}, \ldots, r_{(i+1)k}$. We have that ϕ_n, w_n are exactly ϕ, w modified by function k-ProcessCore, with the exception of the cost of c_n. In fact, $w_n(c_n) = m$, while $w(c_n) = 0$ in function k-ProcessCore. However, for any model I of ϕ_n, $I \models c_i$ implies $\{x_0, \ldots, x_{i \cdot k}\} \subseteq I$, for all $i \in [1..n]$, and therefore $c_n \notin I$. And indeed variable $lower\_bound$ is increased by m in Algorithm 1. This complete our proof. □

Corollary 1. PMRES *is correct also if clauses of the form $c_{i-1} \wedge x_i \rightarrow c_i$ are replaced by clauses of the form $c_i \rightarrow r_i$.*

Proof. Let $C = \{x_0, \ldots, x_n\}$, for some $n \geq 0$. By Theorem 1, Algorithm 1 using 1-ProcessCore is correct. Note that the truth of c_i implies the truth of r_i because of $c_i \rightarrow r_i$, and therefore also of c_{i-1} and x_i because of the constraint, for all $i \in [1..n]$, where $c_0 = x_0$. Hence, clauses $c_i \rightarrow c_{i-1}$ and $c_i \rightarrow x_i$ are implicit in 1-ProcessCore. We complete the proof by noting that, when c_i is false, constraint $c_{i-1} + x_i + \neg c_i + \neg r_i \geq 2$ is equivalent to $c_{i-1} \vee x_i \vee \neg r_i$. □