Propositional Logic
Syntax and Semantics

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Outline

Propositional Logic — Intuition

- Assume basic statements (propositions) to be given
- Make formulas out of them using a fixed set of connectives
- Truth of propositions determines truth of formulas

Example

1. These are beans from the sack
2. These are green beans
3. These are beans from the sack **and** these are green beans

Propositions: 1, 2
Formulas (not being propositions): 3

Knowing that 1 and 2 are true we can conclude that 3 is true!

Well-formed formulas (wff) (1)

Let V be a countable set of propositional variables

Countable means of the same cardinality of natural numbers, or smaller

Example. $V = \{A, B, C, D, A_0, A_1, \ldots\}$

Important!

- No fixed meaning is associated to propositional variables!
- They can mean anything
- Truth value of variables fixed by semantics lateron

The set F_V of propositional formulas (or wff) for V can defined inductively (in the next slides)

Well-formed formulas (wff) (2)

Propositional **variables** are wff

- If $v \in V$ then $v \in F_V$
- Propositional variables are called **atomic formulas**, or (propositional) **atoms**

$\top$ is a wff

- $\top \in F_V$
- $\top$ is called **verum**
- It is always true

$\bot$ is a wff

- $\bot \in F_V$
- $\bot$ is called **falsum**
- It is always false

Well-formed formulas (wff) (3)

The **negation** of a wff is a wff

- If ϕ is a wff then $(\neg\phi)$ is a wff
- If $\phi \in F_V$ then $(\neg\phi) \in F_V$
- $(\neg\phi)$ should always have the opposite truth value of ϕ

Warning! ϕ is a meta-symbol, a placeholder for a wff (not a wff itself)

Well-formed formulas (wff) (4)

The **conjunction** of two wffs is a wff

- If ϕ and ψ are wffs then $(\phi \wedge \psi)$ is a wff
- If $\phi, \psi \in F_V$ then $(\phi \wedge \psi) \in F_V$
- $(\phi \wedge \psi)$ should be true if both ϕ and ψ are true

The **disjunction** of two wffs is a wff

- If ϕ and ψ are wffs then $(\phi \vee \psi)$ is a wff
- If $\phi, \psi \in F_V$ then $(\phi \vee \psi) \in F_V$
- $(\phi \vee \psi)$ should be true if ϕ, ψ or both ϕ and ψ are true
- $\vee$ is also referred to as inclusive or

Warning! ϕ and ψ are meta-symbols, and they can also be equal!

Well-formed formulas (wff) (5)

Implication

- If ϕ and ψ are wffs then $(\phi \to \psi)$ is a wff
- If $\phi, \psi \in F_V$ then $(\phi \to \psi) \in F_V$
- $(\phi \to \psi)$ should be true if ψ is true whenever ϕ is true
- $\to$ is sometimes written as $\supset$

Equivalence

- If ϕ and ψ are wffs then $(\phi \leftrightarrow \psi)$ is a wff
- If $\phi, \psi \in F_V$ then $(\phi \leftrightarrow \psi) \in F_V$
- $(\phi \leftrightarrow \psi)$ should be true if ψ and ϕ have the same truth value

Warning! ϕ and ψ are meta-symbols, and they can also be equal!

Well-formed formulas: Summary

$\phi \in F_V$ if and only if

- $\phi \in V$ or (atoms)
- $\phi = \top$ or (verum)
- $\phi = \bot$ or (falsum)
- $\phi = (\neg\psi)$ where $\psi \in F_V$ or (negation)
- $\phi = (\psi \wedge \psi')$ where $\psi, \psi' \in F_V$ or (conjunction)
- $\phi = (\psi \vee \psi')$ where $\psi, \psi' \in F_V$ or (disjunction)
- $\phi = (\psi \rightarrow \psi')$ where $\psi, \psi' \in F_V$ or (implication)
- $\phi = (\psi \leftrightarrow \psi')$ where $\psi, \psi' \in F_V$ (equivalence)

Warning! ϕ, ψ and ψ' are meta-symbols, and they can also be equal!

Some wffs of $V = \{A, B, C\}$

- A
- $(A \rightarrow \bot)$
- $(A \rightarrow A)$
- $((A \vee B) \leftrightarrow (B \vee A))$
- $((A \vee B) \leftrightarrow (\neg(\top \rightarrow (A \wedge B))))$
- $((\neg(A \rightarrow B)) \rightarrow ((\neg A) \wedge C))$

Equivalent definitions

Formal grammar

Terminals: $V \cup \{\top, \bot\} \cup \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\} \cup \{(,)\}$

Nonterminals: F_V

Start symbol: F_V

Production rules:

- $F_V \longrightarrow v \in V \mid \top \mid \bot$
- $F_V \longrightarrow (\neg F_V)$
- $F_V \longrightarrow (F_V \wedge F_V)$
- $F_V \longrightarrow (F_V \vee F_V)$
- $F_V \longrightarrow (F_V \rightarrow F_V)$
- $F_V \longrightarrow (F_V \leftrightarrow F_V)$

Language elements

- V: propositions
- $\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$: logical connectives
- **(,)**: auxiliary symbols *(parentheses, not comma)*

Eliminating parentheses

- Too many parentheses!!!
- Can we omit a few of them?
- Let us agree on a precedence of connectives (also known as *binding strength*)

Usual assumption

$\neg$ is stronger than
$\wedge$ is stronger than
$\vee$ is stronger than
$\rightarrow$ is stronger than
$\leftrightarrow$

Minimal parentheses

1. $(A \rightarrow \bot)$
2. $(A \rightarrow A)$
3. $((A \vee B) \leftrightarrow (B \vee A))$
4. $((A \vee B) \leftrightarrow (\neg(\top \rightarrow (A \wedge B))))$
5. $(((\neg A) \vee B) \rightarrow ((\neg A) \wedge C))$

Formulas as trees

Every wff can be written as a formula tree

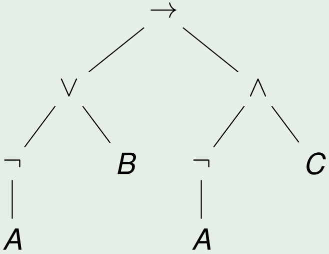
Example

Consider

$$\neg A \vee B \to \neg A \wedge C$$

or equivalently

$$(((\neg A) \vee B) \to ((\neg A) \wedge C))$$



Subformulas (1)

Immediate subformulas of a wff ϕ, denoted *isf*(ϕ)

- If $\phi \in V \cup \{\top, \bot\}$ then $isf(\phi) = \emptyset$
- If $\phi = \neg\psi$ then $isf(\phi) = \{\psi\}$
- If $\phi = \psi \circ \psi'$ for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ then $isf(\phi) = \{\psi, \psi'\}$

Subformulas of a wff ϕ, denoted *sf*(ϕ)

Inductive definition

1. ϕ itself belong to $sf(\phi)$
2. If $\psi \in sf(\phi)$ then $isf(\psi) \subseteq sf(\phi)$

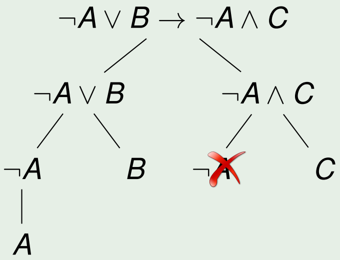
$sf(\phi)$ is the minimal set satisfying conditions 1 and 2

Example

Consider

$$\neg A \vee B \rightarrow \neg A \wedge C$$

Its subformulas are the following:



Semantics

Associate a **meaning** to wffs in a **formal** way

- We could associate "sentences" to variables (e.g. "It is raining.")
- But we are interested only in the truth or falsity of these sentences
- Sentences are informal, truth values can be formalized!

Associate truth values to atoms, truth values for formulas follow

Truth valuations (1)

- Truth values are 1 (true) and 0 (false)
- A (truth) valuation ν is a function

$$\nu\colon V \mapsto \{0, 1\}$$

  - So for each $A \in V$, either $\nu(A) = 1$ or $\nu(A) = 0$, not both

We can extend ν (for atoms) to ν^* (for wffs)

Simple formulas

- $\nu^*(\top) = 1$
- $\nu^*(\bot) = 0$
- If $A \in V$ then $\nu^*(A) = \nu(A)$

Truth valuations (2)

ϕ	ψ	$\neg\phi$
0	0	1
1	0	0
0	1	1
1	1	0

Negation: $\neg\phi$

Should always have the opposite truth value of ϕ

$\nu^*(\neg\phi) = 1$ if and only if $\nu^*(\phi) = 0$

Conjunction: $\phi \wedge \psi$

Should be true if both ϕ and ψ are true

$\nu^*(\phi \wedge \psi) = 1$ iff $\nu^*(\phi) = 1$ and $\nu^*(\psi) = 1$

Disjunction: $\phi \vee \psi$

Should be true if ϕ or ψ are true

$\nu^*(\phi \vee \psi) = 1$ iff $\nu^*(\phi) = 1$ or $\nu^*(\psi) = 1$

Implication: $\phi \to \psi$

Should be true if ψ is true whenever ϕ is true

$\nu^*(\phi \to \psi) = 1$ iff $\nu^*(\phi) = 1$ and $\nu^*(\psi) = 1$, or if $\nu^*(\phi) = 0$

Equivalence: $\phi \leftrightarrow \psi$

Should be true if ϕ has the same truth value as ψ

$\nu^*(\phi \leftrightarrow \psi) = 1$ iff $\nu^*(\phi) = \nu^*(\psi)$

Interpretations

An interpretation I consists exactly of a truth valuation ν

Given a wff ϕ and an interpretation I consisting of valuation ν, let

$$I(\phi) = \nu^*(\phi)$$

- I associates a unique truth value to every formula
- The truth value is the meaning of the formula

I is usually represented as the set of variables interpreted as true:

$$I = \{A \in V \mid I(A) = 1\}$$

Calculating truth values of formulas (1)

Given a formula ϕ and interpretation I, **how** to determine $I(\phi)$?

1. Look at the subformulas of ϕ
2. Work **bottom-up**

Example (For interpretation $I = \{C, A\}$)

Consider

$$\neg A \vee B \rightarrow \neg A \wedge C$$

Its subformulas are the following:

$$\neg A \vee B \rightarrow \neg A \wedge C \; I(\neg A \vee B \rightarrow \neg A \wedge C) = 1$$

$$\neg A \vee B \; I(\neg A \vee B) = 0 \qquad \neg A \wedge C \; I(\neg A \wedge C) = 0$$

$$\neg A \; I(\neg A) = 0 \quad B \; I(B) = 0 \quad \neg A \; I(\neg A) = 0 \quad C \; I(C) = 1$$

$$A \; I(A) = 1 \qquad A \; I(A) = 1$$

Calculating truth values of formulas (2)

A truth table may be useful for calculating truth values for more than one interpretation

Example

Consider

$$\neg A \vee B \to \neg A \wedge C$$

A	B	C	$\neg A$	$\neg A \vee B$	$\neg A \wedge C$	$\neg A \vee B \to \neg A \wedge C$
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	1	0	0
1	1	1	0	1	0	0

Models

1. An interpretation I is a model of a wff ϕ if $I(\phi) = 1$
2. If $I(\phi) = 1$ then I satisifies ϕ
3. If I satisfies ϕ then we write $I \models \phi$

1, 2 and 3 are equivalent, i.e.,

$$I(\phi) = 1 \Leftrightarrow I \text{ is a model of } \phi \Leftrightarrow I \text{ satisfies } \phi \Leftrightarrow I \models \phi$$

- An interpretation I is not a model of a wff ϕ if $I(\phi) = 0$
- If $I(\phi) = 0$ then I does not satisfy ϕ
- If I does not satisfy ϕ then we write $I \not\models \phi$

$$I(\phi) = 0 \Leftrightarrow I \text{ is not a model of } \phi \Leftrightarrow I \text{ does not satisfy } \phi \Leftrightarrow I \not\models \phi$$

Let $V = \{x, y, z\}$ be a set of propositional variables. Show whether the following are well-formed formulas (*not* considering operator preferences) in F_V or not:

1. $(\neg(z \wedge x))$
2. $((x \leftarrow y) \wedge z)$
3. $((x \leftrightarrow (\neg y)))$
4. $(\neg((\neg(\neg z)) \vee (\neg x)))$
5. $((x \leftrightarrow y) \wedge (\neg\neg x))$
6. $((y \wedge x) \leftrightarrow (w \wedge (\neg z)))$
7. $((x \rightarrow (\neg x)) \vee (\neg(y) \rightarrow (\neg z)))$

Let $V = \{x, y, z\}$ be a set of propositional variables.
Draw the parse tree and simplify (according to the standard preferences) as much as possible each of the following formulas:

1. $(\neg(y \land (\neg(z \to (x \lor (\neg y))))))$
2. $(((\neg x) \lor (\neg y)) \leftrightarrow ((\neg y) \land (\neg z)))$
3. $((x \to (\neg y)) \leftrightarrow (\neg(y \land (\neg z))))$
4. $((\neg x) \land ((\neg y) \to (\neg(z \lor (\neg y)))))$

Let $V = \{x, y, z\}$ be a set of propositional variables.
Find the full version (according to the standard preferences) and draw the parse trees of the following formulas:

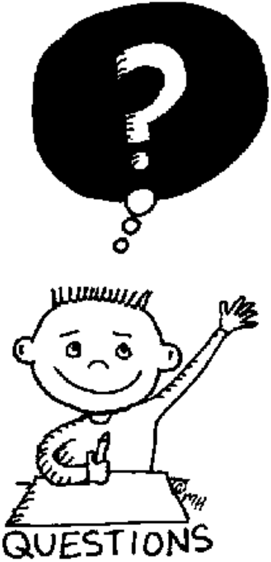
1. $x \vee \neg y \wedge \neg z$
2. $\neg x \vee z \leftrightarrow y \wedge \neg z$
3. $\neg x \leftrightarrow z \rightarrow \neg y \wedge z$
4. $x \leftrightarrow \neg y \rightarrow \neg z \wedge \neg x \vee y$

1. List all subformulas of formulas
   - $x \vee \neg y \wedge \neg z$
   - $\neg x \vee z \leftrightarrow y \wedge \neg z$
2. Given the interpretation $I = \{x, z\}$, decide whether I is a model of

$$x \vee \neg y \wedge \neg z$$

3. Work out the truth table of formula

$$((\neg x) \wedge ((\neg y) \rightarrow (\neg(z \vee (\neg y)))))$$



END OF THE LECTURE