Propositional Logic
Properties

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Outline

Validity and satisfiability (1)

A wff ϕ is

- **valid** if all interpretations are models of ϕ
  - In this case ϕ is also called a **tautology**
- **invalid** if not all interpretations are models of ϕ
- **satisfiable** if there exists an interpretation which is a model of ϕ
- **unsatisfiable** if no interpretation is a model of ϕ
  - In this case ϕ is also called a **contradiction**

Observations

1. Every tautology is satisfiable
2. Every contradiction is invalid
3. The converse of 1 and 2 does not hold in general
4. Tautologies and contradictions form disjoint sets

Validity and satisfiability (2)

Example

A	B	C	$\neg A$	$\neg A \vee B$	$\neg A \wedge C$	$\neg A \vee B \rightarrow \neg A \wedge C$
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	1	0	0
1	1	1	0	1	0	0

- $\neg A \vee B \rightarrow \neg A \wedge C$ is invalid
- $\neg A \vee B \rightarrow \neg A \wedge C$ is satisfiable
- $\neg A \vee B \rightarrow \neg A \wedge C$ is neither a tautology nor a contradiction

Validity and satisfiability (3)

Examples

Identify tautologies and contradictions among the following formulas:

- $\phi \to \phi$ — tautology
- $\phi \wedge \neg\phi$ — contradiction
- $\phi \vee \neg\phi$ — tautology
- $\phi \vee \top$ — tautology
- $\phi \wedge \bot$ — contradiction

Equivalence (1)

Two wffs ϕ, ψ are **equivalent** if

- for each interpretation *I* we have $I(\phi) = I(\psi)$, i.e.
- ϕ and ψ have the same models, i.e.
- $\phi \leftrightarrow \psi$ is valid

Equivalence of ϕ and ψ is denoted as $\phi \equiv \psi$

Observations

- A wff ϕ is a tautology if and only if $\phi \equiv \top$
- A wff ϕ is a contradiction if and only if $\phi \equiv \bot$

Equivalence (2)

Example

- $\phi \vee \psi \equiv \psi \vee \phi$ (commutativity)
- $\phi \wedge \psi \equiv \psi \wedge \phi$ (commutativity)
- $\phi \leftrightarrow \psi \equiv \psi \leftrightarrow \phi$ (commutativity)
- $\phi \vee \phi \equiv \phi$ (idempotence)
- $\phi \wedge \phi \equiv \phi$ (idempotence)
- $\phi \vee \top \equiv \top$
- $\phi \wedge \bot \equiv \bot$

Example

- $\phi \vee \bot \equiv \phi$ (neutrality)
- $\phi \wedge \top \equiv \phi$ (neutrality)
- $\phi \vee \neg\phi \equiv \top$
- $\phi \wedge \neg\phi \equiv \bot$
- $\neg\neg\phi \equiv \phi$
- $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$
- $\phi \rightarrow \psi \equiv \neg\psi \rightarrow \neg\phi$ (contraposition)

Equivalence (4)

Example

- $\neg(\phi \vee \psi) \equiv \neg\psi \wedge \neg\phi$ (De Morgan)
- $\neg(\phi \wedge \psi) \equiv \neg\psi \vee \neg\phi$ (De Morgan)
- $\phi \vee (\psi \vee \gamma) \equiv (\phi \vee \psi) \vee \gamma$ (associativity)
- $\phi \wedge (\psi \wedge \gamma) \equiv (\phi \wedge \psi) \wedge \gamma$ (associativity)
- $\phi \wedge (\psi \vee \gamma) \equiv (\phi \wedge \psi) \vee (\phi \wedge \gamma)$ (distributivity)
- $\phi \vee (\psi \wedge \gamma) \equiv (\phi \vee \psi) \wedge (\phi \vee \gamma)$ (distributivity)
- $\phi \wedge (\phi \vee \psi) \equiv \phi$ (absorption)
- $\phi \vee (\phi \wedge \psi) \equiv \phi$ (absorption)

Entailment

- So far interpretations assigned truth values to wffs
- Can we extend interpretations to set of wffs?

Definition (Model of a set of wffs)

Given an interpretation I and a set of wffs Γ,
$I(\Gamma) = 1$ iff for each $\phi \in \Gamma$ we have $I(\phi) = 1$

Example

Consider $I = \{A, B, D\}$

- $I(\{A \leftrightarrow B \wedge D, \neg C\}) = 1$
- $I(\{A, B, C \rightarrow D, B \leftrightarrow C\}) = 0$

Warning!

- A set Γ of wffs could be seen as the conjunction of its formulas
- But this conjunction is not a wff in general!
- Actually, it is a wff only if Γ has finite cardinality

Entailment

- Can we represent an interpretation I as a set Γ of wffs such that I is the only model of Γ? Yes!

For $I = \{T_1, T_2, \ldots\}$, consider $\Gamma = \{T_1, T_2, \ldots, \neg F_1, \neg F_2, \ldots\}$, where $F_1, F_2, \ldots$ are the false variables according to I

- We can thus define $\Gamma \models \phi$ for a set Γ of wffs and a wff ϕ

Definition

$\Gamma \models \phi$ if each model of Γ is also a model of ϕ

- If $\Gamma \models \phi$ then we say that Γ entails ϕ
- We also say that ϕ is a logical consequence of Γ

$\models \phi$ if and only if ϕ is a tautology

Entailment

Example

- $\{\phi\} \cup \Gamma \models \phi$
- $\{\phi \wedge \psi\} \models \phi \to \psi$
- $\{\neg\phi\} \models \phi \to \psi$
- $\{\phi, \psi\} \models \phi \to \psi$
- $\{\phi\} \models \phi \vee \psi$

Note: We will often omit curly brackets

Theorem (Monotonicity)

If $\Gamma \models \phi$ *then* $\Gamma, \psi \models \phi$

Deduction Theorem

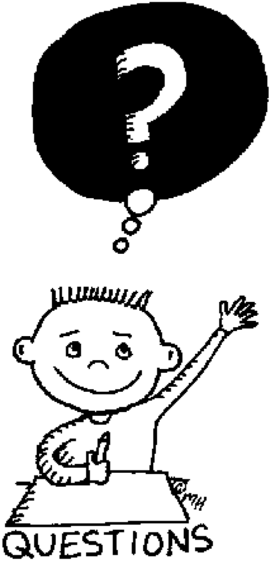
$\Gamma, \phi \models \psi$ if and only if $\Gamma \models \phi \to \psi$

Contraposition Theorem

$\Gamma \models \phi \to \psi$ if and only if $\Gamma, \neg\psi \models \neg\phi$

Exercises (1)

1. Decide whether formula
$$x \vee \neg y \wedge \neg z$$
is satisfiable
2. Decide whether formula
$$\neg x \vee z \leftrightarrow y \wedge \neg z$$
is a tautology
3. Decide whether formula
$$x \leftrightarrow \neg y \rightarrow \neg z \wedge \neg x \vee y$$
is a contradiction



END OF THE LECTURE