First-order logic
Syntax and semantics

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Outline

Propositional logic: is it enough? (1)

We want to represent:

- Socrates is a human
- Humans are mortal

From this, we want to draw the conclusion:

- Socrates is mortal

In propositional logic

- Variable SH for “Socrates is a human”
- Variable HM for “Humans are mortal”
- Variable SM for “Socrates is mortal”
- We define $\Gamma := \{SH, SH \to SM\}$
- From this, we can draw the conclusion: $\Gamma \models SM$

Where is *HM*?!?

This is not what we wanted to express!

Propositional logic: is it enough? (2)

“Humans are mortal” is not an atom!

- Mario is a human
- Humans are mortal
- Mario is mortal

- We talk about objects, not about propositions!

But...

There is no concept of object in propositional logic

Towards first-order logic

- Start from propositional logic
- Atoms are no longer indivisible
- Compose atoms from:
  - terms, representing objects, and
  - predicates, i.e., statements about terms

Example

- Socrates is a term
- Mortal is a predicate

Functions

Socrates' father

- Represents exactly one object
- But we refer to it from another object

- Function symbols map some objects to other objects

Variables

Humans are mortal

- If some object is human then it is mortal
- We are not referring to any specific object

- We need a variable ranging over objects
- **Note:** Such variables are completely different from propositional variables!
- Think about them as object variables

Quantifiers

Humans are mortal

- Actually wants to express all humans are mortal
- And not “some humans are mortal”
- And certainly not “no humans are mortal”

- Quantifiers express
  - for all objects, or
  - some object exists

Function symbols and constants

- Let $\mathcal{F}$ be a countable set of function symbols
- An arity (nonnegative integer) is associated with each function symbol
- Function symbols with arity 0 are constants (i.e., objects)

Examples

- *socrates* (arity 0)
- *father* (arity 1)
- *firstSonOf* (arity 2)
- *supercalifragilistichespiralidoso*287 (arity 6)

Variables

- Let $\mathcal{V}$ be a countable set of (object) variables
- **Note:** $\mathcal{V}$ and $\mathcal{F}$ are disjoint!

Examples

- x, y, z
- *_human*
- *_xiknve*

Terms (1)

Build terms from function symbols and variables, respecting arities

Terms: Inductive definition

- If v is a variable symbol then v is a term
- If f is a function symbol of arity 0 then f is a term
- If f is a function symbol of arity $n > 0$, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term

Ground terms

Terms not containing any variable!

Terms (2)

Examples

- *socrates*
- *_xiknve*
- *father*(*socrates*)
- *father*(*father*(*socrates*))
- *father*(*father*(*father*(*socrates*)))
- *firstSon*(*father*(*socrates*), *_xiknve*)
- *supercalifragilistichespiralidoso287*(x, y, z, *socrates*, *_xiknve*, x)

Intuitively, each term represents an object

Predicate symbols

- Let $\mathcal{P}$ be a countable set of predicate symbols
- An arity is associated with each predicate symbol

Examples

- *Human* (arity 1)
- *Mortal* (arity 1)
- *Married* (arity 2)
- *Yggdrasil* (arity 18)

Atoms

Atomic formula, or atom

A structure of the form $P(t_1, \ldots, t_n)$, where

- P is a predicate symbol of arity $n \geq 0$
- $t_1, \ldots, t_n$ are terms

Examples

- $\mathit{Human}(\mathit{socrates})$
- $\mathit{Mortal}(\mathit{father}(\_\mathit{xiknve}))$
- $\mathit{Married}(\mathit{socrates}, x)$

Predicates with arity 0 are like propositional variables!

Formulas

Similar to propositional logic with different atoms!

ϕ is a wff if and only if

- ϕ is an atomic formula, or
- $\phi = \top$, or
- $\phi = \bot$, or
- $\phi = (\neg\psi)$ where ψ is a formula, or
- $\phi = (\psi \wedge \psi')$ where ψ, ψ' are formulas, or
- $\phi = (\psi \vee \psi')$ where ψ, ψ' are formulas, or
- $\phi = (\psi \rightarrow \psi')$ where ψ, ψ' are formulas, or
- $\phi = (\psi \leftrightarrow \psi')$ where ψ, ψ' are formulas, or
- $\phi = (\forall x\ \psi)$ where x is a variable and ψ is a formula, or
- $\phi = (\exists x\ \psi)$ where x is a variable and ψ is a formula

Quantified variables

- $\forall$ is the universal quantifier
- $\exists$ is the existential quantifier
- In $\forall x\ \psi$ and $\exists x\ \psi$
  - ψ is the scope of x
  - x is bound in ψ
- Variables which are not bound in a formula are free
- Formulas without free variables are closed, also called sentences

Binding strength

As for propositional logic, we usually minimize parentheses

- $\forall$, $\exists$, and $\neg$ have the same binding strength
- Then come $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, in this order

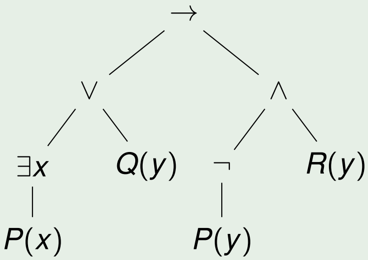
Examples

- $Human(socrates)$
- $Mortal(socrates)$
- $(\forall x\ (Human(x) \rightarrow Mortal(x)))$
- $(\forall x\ (Human(firstSonOf(socrates, x)) \rightarrow Married(socrates, x)))$
- $(\forall x\ ((\exists y\ Human(firstSonOf(x, y))) \rightarrow (\exists z\ Married(x, z))))$
- $((\exists y\ Human(firstSonOf(x, y))) \rightarrow (\exists z\ Married(x, z)))$

Formulas as trees

Every formula can be written as a formula tree

$\exists x\, P(x) \lor Q(y) \to \neg P(y) \land R(y)$



Similarly, we can define (immediate) subformulas

- How to determine the scope of a variable?
- How to determine bound variables?

Examples

Example 1

1. $N(o)$
2. $\forall x\ (N(x) \rightarrow N(s(x)))$
3. $\forall x\ \forall y\ (\neg E(x, y) \rightarrow \neg E(s(x), s(y)))$
4. $\forall x\ \neg E(s(x), o)$

Example 2

1. $\forall x\ \neg E(s(x), o)$
2. $\forall x\ \forall y\ (E(s(x), s(y)) \rightarrow E(x, y))$
3. $\forall x\ E(a(x, o), x)$
4. $\forall x\ \forall y\ E(a(x, s(y)), s(a(x, y)))$
5. $\forall x\ E(m(x, o), o)$
6. $\forall x\ \forall y\ E(m(x, s(y)), a(m(x, y), x))$

Free variables

- Given a term t, the set $\mathit{free}(t)$ of its free variables is defined as:
  - $\mathit{free}(t) = \{x\}$ if t is a variable x
  - $\mathit{free}(t) = \emptyset$ if t is a constant
  - $\mathit{free}(t) = \bigcup_{i=1}^{n} \mathit{free}(t_i)$ if t is $f(t_1, \ldots, t_n)$
- Given a formula ϕ, the set $\mathit{free}(\phi)$ of its free variables is defined as:
  - $\mathit{free}(\phi) = \bigcup_{i=1}^{n} \mathit{free}(t_i)$ if ϕ is an atom $P(t_1, \ldots, t_n)$
  - $\mathit{free}(\phi) = \emptyset$ if ϕ is $\top$ or $\bot$
  - $\mathit{free}(\phi) = \mathit{free}(\psi)$ if ϕ is $\neg\psi$
  - $\mathit{free}(\phi) = \mathit{free}(\psi) \cup \mathit{free}(\psi')$ if ϕ is $\psi \odot \psi'$ for $\odot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
  - $\mathit{free}(\phi) = \mathit{free}(\psi) \setminus \{x\}$ if ϕ is $\forall x\ \psi$ or $\exists x\ \psi$

Substitute variable x by term t in term s, denoted $s[t/x]$:

- $s[t/x] = t$ if s is a variable and $s = x$
- $s[t/x] = s$ if s is a variable and $s \neq x$
- $s[t/x] = f(t_1[t/x], \ldots, t_n[t/x])$ if $s = f(t_1, \ldots, t_n)$

Substitute variable x by term t in a formula ϕ, denoted $\phi[t/x]$:

- $\phi[t/x] = \phi$ if ϕ is $\top$ or $\bot$
- $\phi[t/x] = P(t_1[t/x], \ldots, t_n[t/x])$ if ϕ is an atom $P(t_1, \ldots, t_n)$
- $\phi[t/x] = \neg\psi[t/x]$ if $\phi = \neg\psi$
- $\phi[t/x] = \psi[t/x] \odot \psi'[t/x]$ if $\phi = \psi \odot \psi'$ for $\odot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
- if $\phi = \mathcal{Q}y\ \psi$ for $\mathcal{Q} \in \{\forall, \exists\}$ then
  - $\phi[t/x] = \mathcal{Q}y\ \psi[t/x]$ if $x \neq y$ and $y \notin \mathit{free}(t)$
  - $\phi[t/x] = \mathcal{Q}z\ \psi[z/y][t/x]$ if $x \neq y$ and $y \in \mathit{free}(t)$, where z is a variable not occurring in ψ and t
  - $\phi[t/x] = \mathcal{Q}y\ \psi$ if $x = y$

Term substitution (2)

- $t[t_1/x_1, \ldots, t_n/x_n]$ denotes the simultaneous substitution of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$ in the term t
- $\phi[t_1/x_1, \ldots, t_n/x_n]$ denotes the simultaneous substitution of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$ in the formula ϕ

Example

Compare the followings:

1. $A(x, y)[y/x, x/y]$
2. $A(x, y)[y/x][x/y]$
3. $A(x, y)[x/y][y/x]$

Formula substitution

- In propositional logic we replaced atoms by formulas
- Can we do the same in first-order logic?
- Yes, we can!

Definition

For wffs γ, ϕ and atom *A*, let $\gamma[\phi/A]$ denote the formula in which ϕ replaces all occurrences of *A*

Semantics — Requirements

Domain

- We want to speak about objects
- Each term represents an object
- We need a set of objects called domain (or universe) of discourse

Interpretation function

- A constant symbol should be associated to an object in the domain
- A function symbol of arity n should be associated to a function mapping each n-tuple of objects to an object
- A predicate of arity n should be associated to a function mapping each n-tuple of objects to a Boolean value
- We need an interpretation function for this purpose

Structures

Definition

A structure is a pair $\mathcal{A} = (D_{\mathcal{A}}, I_{\mathcal{A}})$, where

- $D_{\mathcal{A}}$ is a nonempty set, called domain
- $I_{\mathcal{A}}$ is an interpretation function, i.e., a function associating
  - each constant symbol c with an object $I_{\mathcal{A}}(c) \in D_{\mathcal{A}}$
  - each function symbol f of arity n with a function

$$I_{\mathcal{A}}(f) : D_{\mathcal{A}}^n \to D_{\mathcal{A}}$$

  - each predicate symbol P of arity n with a relation

$$I_{\mathcal{A}}(P) : D_{\mathcal{A}}^n \to \{0, 1\}$$

Notation. We will usually write $c^{\mathcal{A}}$, $f^{\mathcal{A}}$ and $P^{\mathcal{A}}$ instead of $I_{\mathcal{A}}(c)$, $I_{\mathcal{A}}(f)$ and $I_{\mathcal{A}}(P)$, respectively

Observation. Constants are actually associated with functions of arity 0

Variable assignments

- What about (free) variables?
- We need a variable assignment, also called environment

Definition

A variable assignment $\xi^{\mathcal{A}}$ for a structure $\mathcal{A}$ is a function from the set of variables to the domain, i.e.,

$$\xi^{\mathcal{A}} : \mathcal{V} \rightarrow D_{\mathcal{A}}$$

Notation

We extend substitutions to variable assignments.
Let $\xi^{\mathcal{A}}$ be a variable assignment, x be a variable and d be an object in $D_{\mathcal{A}}$.

- $\xi^{\mathcal{A}}[d/x]$ is the variable assignment such that
  - $\xi^{\mathcal{A}}[d/x](y) = d$ if $y = x$
  - $\xi^{\mathcal{A}}[d/x](y) = \xi^{\mathcal{A}}(y)$ if $y \neq x$

Valuation

Interpretation

An interpretation for a first-order language is a pair $(\mathcal{A}, \xi^{\mathcal{A}})$, where

- $\mathcal{A} = (D_{\mathcal{A}}, I_{\mathcal{A}})$ is a structure
- $\xi^{\mathcal{A}}$ is a variable assignment for $\mathcal{A}$

- The valuation $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t)$ of a term t wrt $(\mathcal{A}, \xi^{\mathcal{A}})$ is
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t) = \xi^{\mathcal{A}}(t)$ if t is a variable
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t) = f^{\mathcal{A}}(\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t_1), \ldots, \nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t_n))$ if t is $f(t_1, \ldots, t_n)$
- The valuation of formulas is also defined inductively
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\top) = 1$ and $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\bot) = 0$
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(P(t_1, \ldots, t_n)) = P^{\mathcal{A}}(\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t_1), \ldots, \nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t_n))$
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\neg\phi) = \neg\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\phi)$ using propositional logic
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\phi \odot \psi) = \nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\phi) \odot \nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\psi)$ using propositional logic, for $\odot \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\forall x\ \phi) = 1$ iff $\nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/x])}(\phi) = 1$ for all $d \in D_{\mathcal{A}}$
  - $\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\exists x\ \phi) = 1$ iff $\nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/x])}(\phi) = 1$ for some $d \in D_{\mathcal{A}}$

Example 1

1. $N(o)$
2. $\forall x\ (N(x) \rightarrow N(s(x)))$
3. $\forall x\ \forall y\ (\neg E(x, y) \rightarrow \neg E(s(x), s(y)))$
4. $\forall x\ \neg E(s(x), o)$

- $D_{\mathcal{A}} = \mathbb{N}$ (natural numbers, including 0)
- $o^{\mathcal{A}} = 0$
- $s^{\mathcal{A}}$: Successor of a number
- $N^{\mathcal{A}}$: True for natural numbers
- $E^{\mathcal{A}}$: True for equal numbers
- $\xi^{\mathcal{A}}$? Not relevant because there are no free variables

Example 2

1. $\forall x \, \neg E(s(x), o)$
2. $\forall x \, \forall y \, (E(s(x), s(y)) \rightarrow E(x, y))$
3. $\forall x \, E(a(x, o), x)$
4. $\forall x \, \forall y \, E(a(x, s(y)), s(a(x, y)))$
5. $\forall x \, E(m(x, o), o)$
6. $\forall x \, \forall y \, E(m(x, s(y)), a(m(x, y), x))$

- $D_{\mathcal{A}} = \mathbb{N}$ (natural numbers, including 0)
- $o^{\mathcal{A}} = 0$
- $s^{\mathcal{A}}$: Successor of a number
- $a^{\mathcal{A}}$: Sum of two numbers
- $m^{\mathcal{A}}$: Product of two numbers
- $E^{\mathcal{A}}$: True for equal numbers
- $\xi^{\mathcal{A}}$? Not relevant because there are no free variables

Socrates example

1. $Human(socrates)$
2. $\forall x\ (Human(x) \rightarrow Mortal(x))$
3. $Mortal(socrates)$

- Can we conclude 3 from 1 and 2?
- Not yet! We must still define the notions of model and entailment

Peano (first-order) axioms

1. $N(0)$
2. $\forall x\, E(x,x)$
3. $\forall x\, \forall y\, (E(x,y) \rightarrow E(y,x))$
4. $\forall x\, \forall y\, \forall z\, (E(x,y) \wedge E(y,z) \rightarrow E(x,z))$
5. $\forall x\, \forall y\, (N(x) \wedge E(x,y) \rightarrow N(y))$
6. $\forall x\, (N(x) \rightarrow N(s(x)))$
7. $\forall x\, \neg E(s(x),0)$
8. $\forall x\, \forall y\, (E(s(x),s(y)) \rightarrow E(x,y))$
9. If P is a predicate of arity 1 then
    - $(P(0) \wedge \forall x\, (P(x) \rightarrow P(s(x)))) \rightarrow \forall x\, P(x)$

Simple observations

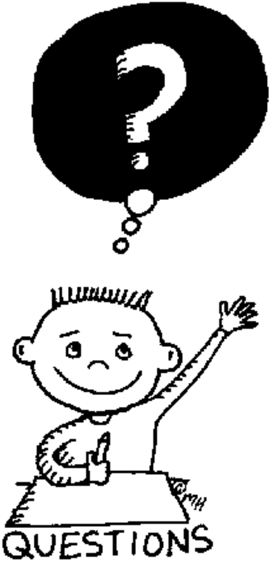
Lemma

Let $\mathcal{A}$ be a structure, and $\xi_1^{\mathcal{A}}, \xi_2^{\mathcal{A}}$ be variable assignments such that $\xi_1^{\mathcal{A}}(x_i) = \xi_2^{\mathcal{A}}(x_i)$ for $i = 1, \ldots, n$.

- If t is a term such that $\mathit{free}(t) = \{x_1, \ldots, x_n\}$ then $\nu^{(\mathcal{A}, \xi_1^{\mathcal{A}})}(t) = \nu^{(\mathcal{A}, \xi_2^{\mathcal{A}})}(t)$
- If ϕ is a formula such that $\mathit{free}(\phi) = \{x_1, \ldots, x_n\}$ then $\nu^{(\mathcal{A}, \xi_1^{\mathcal{A}})}(\phi) = \nu^{(\mathcal{A}, \xi_2^{\mathcal{A}})}(\phi)$

Exercises

1. Find the free variables in the following wffs
   1. $\forall y \exists x\, A(x, y) \rightarrow B(x, y)$
   2. $\exists x \exists y (A(x, y) \rightarrow B(x, y))$
   3. $\neg \forall y \exists x\, A(y) \rightarrow (B(x, y) \wedge \forall z\, C(x, z))$
   4. $\exists x \exists y (A(x, y) \rightarrow B(x)) \rightarrow \forall z\, C(z) \vee D(z)$
2. For each formula ϕ above
   1. find an interpretation valuating ϕ as 0
   2. find an interpretation valuating ϕ as 1
   3. find a structure $\mathcal{A}$ such that $\nu^{(\mathcal{A}, \xi^{\mathcal{A}})}(\phi) = 1$ for every $\xi^{\mathcal{A}}$



END OF THE LECTURE