First-order logic
Semantic notions and sequent calculus

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Outline

Model

Let ϕ be a wff.

1. ϕ is satisfied in a structure $\mathcal{A}$ wrt a variable assignment $\xi^{\mathcal{A}}$, denoted $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$, if $\nu^{(\mathcal{A}, \xi^{\mathcal{A}})}(\phi) = 1$
2. ϕ is satisfiable in $\mathcal{A}$ if there is $\xi^{\mathcal{A}}$ such that $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$
3. $\mathcal{A}$ is a model of ϕ, denoted $\mathcal{A} \models \phi$, if $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$ for every $\xi^{\mathcal{A}}$
4. ϕ is satisfiable if there is a structure $\mathcal{A}$ such that ϕ is satisfiable in $\mathcal{A}$
5. ϕ is valid, denoted $\models \phi$, if $\mathcal{A} \models \phi$ for every structure $\mathcal{A}$

Entailment

Let Γ be a set of wffs.

1. Γ is satisfied in a structure $\mathcal{A}$ wrt $\xi^{\mathcal{A}}$ if $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$ for each $\phi \in \Gamma$
2. $\mathcal{A}$ is a model of Γ if $\mathcal{A} \models \phi$ for each $\phi \in \Gamma$

Let Γ be a set of wffs and ϕ a wff.

- ϕ is a semantic (or logic) consequence of Γ, denoted $\Gamma \models \phi$, if $(\mathcal{A}, \xi^{\mathcal{A}}) \models \phi$ whenever Γ is satisfied in $\mathcal{A}$ wrt $\xi^{\mathcal{A}}$, i.e., whenever $(\mathcal{A}, \xi^{\mathcal{A}}) \models \psi$ for each $\psi \in \Gamma$

Socrates example

1. $Human(socrates)$
2. $\forall x \ (Human(x) \rightarrow Mortal(x))$
3. $Mortal(socrates)$

- Can we conclude 3 from 1 and 2?
- Yes, now we can!

$$1, 2 \models 3$$

Closures

Definition

Let ϕ be a wff such that *free*$(\phi) = \{x_1, \ldots, x_n\}$.

- $Cl(\phi) = \forall x_1 \cdots \forall x_n \, \phi$ is the universal closure of ϕ
- $Ex(\phi) = \exists x_1 \cdots \exists x_n \, \phi$ is the existential closure of ϕ

Theorems

- $\mathcal{A} \models \phi$ if and only if $\mathcal{A} \models Cl(\phi)$
- $\models \phi$ if and only if $\models Cl(\phi)$
- ϕ is satisfiable if and only if $Ex(\phi)$ is satisfiable

Equivalence (1)

Definition

$\phi \equiv \psi$ if for every $(\mathcal{A}, \xi^{\mathcal{A}})$ we have

$$\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\phi) = \nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\psi)$$

- All equivalences proved for propositional logic hold for first-order formulas
  - Commutativity, idempotence, neutrality, contraposition
  - De Morgan, associativity, distributivity, absorption
  - Reductions of computational problems
  - Substitution Theorem
- Can we do more?

Equivalence (2)

Substitution Lemma

Let $(\mathcal{A}, \xi^{\mathcal{A}})$ be an interpretation, x, y variables, and d an object in $D_{\mathcal{A}}$.

1. If y does not occur in a term t then

$$\nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/x])}(t) = \nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/y])}(t[y/x])$$

2. If y does not occur in a formula ϕ then

$$\nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/x])}(\phi) = \nu^{(\mathcal{A},\xi^{\mathcal{A}}[d/y])}(\phi[y/x])$$

Equivalence (3)

- $\exists x\ \phi \equiv \exists z\ \phi[z/x]$ if z does not occur in ϕ (rename)
- $\forall x\ \phi \equiv \forall z\ \phi[z/x]$ if z does not occur in ϕ (rename)
- $\neg\forall x\ \phi \equiv \exists x\ \neg\phi$ (De Morgan)
- $\neg\exists x\ \phi \equiv \forall x\ \neg\phi$ (De Morgan)
- $\forall x\ \phi \equiv \neg\exists x\ \neg\phi$ (De Morgan)
- $\exists x\ \phi \equiv \neg\forall x\ \neg\phi$ (De Morgan)
- $\forall x \forall y\ \phi \equiv \forall y \forall x\ \phi$ (exchange)
- $\exists x \exists y\ \phi \equiv \exists y \exists x\ \phi$ (exchange)
- $\forall x\ \phi \equiv \phi$ if $x \notin \mathit{free}(\phi)$
- $\exists x\ \phi \equiv \phi$ if $x \notin \mathit{free}(\phi)$

Equivalence (4)

- $\forall x\ (\phi \wedge \psi) \equiv \forall x\ \phi \wedge \forall x\ \psi$ (distributivity)
- $\exists x\ (\phi \vee \psi) \equiv \exists x\ \phi \vee \exists x\ \psi$ (distributivity)
- $\forall x\ (\phi \vee \psi) \equiv \forall x\ \phi \vee \psi$ if $x \notin \mathit{free}(\psi)$ (distributivity)
- $\exists x\ (\phi \wedge \psi) \equiv \exists x\ \phi \wedge \psi$ if $x \notin \mathit{free}(\psi)$ (distributivity)
- $\forall x\ (\phi \vee \psi) \not\equiv \forall x\ \phi \vee \forall x\ \psi$ if $x \in \mathit{free}(\psi)$
- $\exists x\ (\phi \wedge \psi) \not\equiv \exists x\ \phi \wedge \exists x\ \psi$ if $x \in \mathit{free}(\psi)$
- $\mathcal{Q}_1 x\ \phi \vee \mathcal{Q}_2 x\ \psi \equiv \mathcal{Q}_1 x \mathcal{Q}_2 z\ (\phi \vee \psi[z/x])$ if $\mathcal{Q}_1, \mathcal{Q}_2 \in \{\exists, \forall\}$ and $z \notin \mathit{free}(\phi) \cup \mathit{free}(\psi)$
- $\mathcal{Q}_1 x\ \phi \wedge \mathcal{Q}_2 x\ \psi \equiv \mathcal{Q}_1 x \mathcal{Q}_2 z\ (\phi \wedge \psi[z/x])$ if $\mathcal{Q}_1, \mathcal{Q}_2 \in \{\exists, \forall\}$ and $z \notin \mathit{free}(\phi) \cup \mathit{free}(\psi)$
- $\forall x\ \phi \rightarrow \psi \equiv \exists x\ (\phi \rightarrow \psi)$ if $x \notin \mathit{free}(\psi)$
- $\exists x\ \phi \rightarrow \psi \equiv \forall x\ (\phi \rightarrow \psi)$ if $x \notin \mathit{free}(\psi)$
- $\phi \rightarrow \exists x\ \psi \equiv \exists x\ (\phi \rightarrow \psi)$ if $x \notin \mathit{free}(\phi)$
- $\phi \rightarrow \forall x\ \psi \equiv \forall x\ (\phi \rightarrow \psi)$ if $x \notin \mathit{free}(\phi)$

Sequent Calculus for first-order logic

- Just add rules for quantifiers

Logical rules for quantifiers

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x\, A \vdash \Delta}\,(\forall l) \qquad \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x\, A}\,(\forall r)$$

$$\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x\, A \vdash \Delta}\,(\exists l) \qquad \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x\, A}\,(\exists r)$$

where y is not free in the bottom sequents of $(\forall r)$ and $(\exists l)$

1. Given the formula

$$\exists x \forall y \; P(g(x,y), f(x), g(y,z))$$

   define one structure which is a model and
   one which is not a model
2. Given the formulas
   1. $\forall x \; R(x,x)$
   2. $\forall x \forall y \; (R(x,y) \rightarrow R(y,x))$
   3. $\forall x \forall y \forall z \; ((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$

   show that no formula is entailed by the other two formulas.

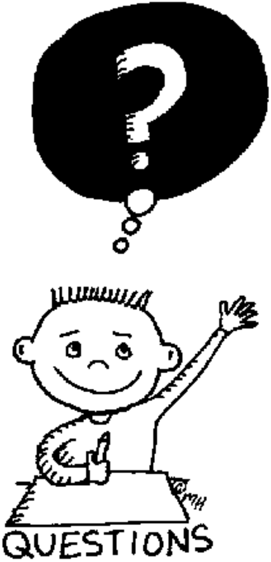
   An easy way of showing this is by defining three structures: The first structure should satisfy formulas 1 and 2, but not 3; the second should satisfy 1 and 3, but not 2; the third should satisfy 2 ad 3, but not 1.

1. Find a model for the following set of wffs:
   - $\forall x\ R(x, g(x))$
   - $\forall y\ \neg R(y, y)$
   - $\forall x \forall y \forall z\ ((R(x, y) \land R(y, z)) \to R(x, z))$

   Then, show that no model for this set of wffs has a finite domain.
2. Show the followings using the sequent calculus:
   1. $\models \exists x\ R(x, x) \to \exists y\ R(y, y)$
   2. $\models \forall x\ R(x, x) \to \neg\exists y\ \neg R(y, y)$
   3. $\models \neg\exists x\ \neg R(x, x) \to \forall y\ R(y, y)$
   4. $\models \neg\exists y \forall x\ (\neg P(x, x) \leftrightarrow P(y, x))$
   5. $\neg\forall x \phi \equiv \exists x \neg\phi$

Prove the followings or provide interpretations proving their falsity:

1. $\forall x\, A(x) \vdash \exists x\, A(x)$
2. $\forall x\, (A(x) \wedge B(x)) \vdash \forall x\, A(x) \wedge \forall x\, B(x)$
3. $\neg\exists x\, A(x) \vdash \exists x\, \neg A(x)$
4. $\forall x \forall y\, C(x, y) \vdash \forall y \forall x\, C(x, y)$
5. $\exists x \forall y\, C(x, y) \vdash \forall y \exists x\, C(x, y)$
6. $\forall x \forall y\, (C(x, y) \rightarrow \neg C(y, x)) \vdash \forall x\, \neg C(x, x)$
7. $\forall x\, (\exists y\, C(x, y) \rightarrow A(x)) \vdash \forall x \exists y\, (C(x, y) \rightarrow A(x))$
8. $\forall x\, (\forall y\, C(x, y) \rightarrow A(x)) \vdash \forall x \exists y\, (C(x, y) \rightarrow A(x))$
9. $\forall x\, (\exists y\, C(x, y) \rightarrow A(x)) \vdash \forall x \forall y\, (C(x, y) \rightarrow A(x))$
10. $\forall x\, (\forall y\, C(x, y) \rightarrow A(x)) \vdash \forall x \forall y\, (C(x, y) \rightarrow A(x))$



END OF THE LECTURE