First-order logic
Normal forms and Herbrand theory

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Outline

Prenex Normal Form

Formulas of the following type are in Prenex Normal Form:

$$\mathcal{Q}_1 x_1 \ \cdots \ \mathcal{Q}_n \, x_n \, \phi$$

where

- $\mathcal{Q}_i \in \{\forall, \exists\}$ for $i = 1, \ldots, n$
  - $\mathcal{Q}_1 x_1 \ \cdots \ \mathcal{Q}_n x_n$ is called quantifier prefix
- ϕ is a quantifier-free formula
  - ϕ is called matrix

Theorem

For each wff ϕ there is an equivalent ϕ^{PNF} in prenex normal form.

Intuitively, move quantifiers outside, i.e., up on the tree structure

Negation, Conjunctive and Disjunctive Normal Form

Negation Normal Form

1. Compute the prenex normal form
2. Compute the negation normal form of the matrix

Conjunctive Normal Form

1. Compute the prenex normal form
2. Compute the conjunctive normal form of the matrix

Disjunctive Normal Form

1. Compute the prenex normal form
2. Compute the disjunctive normal form of the matrix

Skolemization (1)

- Can we further simplify the structure of a wff?
- Can we remove existential quantifiers?
- Yes, by introducing Skolem functions!
- However, equivalence will not hold anymore
- Still we will preserve satisfiability!

Intuition

Consider $\exists x\ A(x)$

- There is x such that $A(x)$ is true
- Replace x with a fresh constant c
- The Skolem formula will be $A(c)$

Consider $\forall x \exists y\ B(x, y)$

- Can we replace y by a constant d?
  - $\mathcal{A}$ such that $D_{\mathcal{A}} = \mathbb{N}^{+}$ and $B^{\mathcal{A}} = \{(x, y) \mid x \text{ divides } y\}$
  - Is d a multiple of every natural number?
- Let's replace y by a function $f(x)$

Skolemization (2)

Skolem Normal Form

Let ϕ be a wff. ϕ^{SNF} is obtained as follows:

1. Compute the prenex normal form of ϕ

$$\gamma := \mathcal{Q}_1 x_1 \ \cdots \ \mathcal{Q}_n \, x_n \, \psi$$

2. If γ has no existential quantifier then stop
3. Let $i \in \{1, \ldots, n\}$ be the first existential quantifier
4. Modify γ:
   - Remove the quantifier $\mathcal{Q}_i x_i$
   - Replace x_i by $f(x_1, \ldots, x_{i-1})$
     where f is a fresh function symbol of arity $i - 1$
5. Go to 2

Functions introduced by the Skolemization are called Skolem functions

Theorem

Let ϕ be a wff.
ϕ is satisfiable if and only if ϕ^{SNF} is satisfiable.

Corollary

Let ϕ be a wff.
ϕ is satisfiable if and only if $Ex(\phi)^{SNF}$ is satisfiable.

Let's consider only closed formulas!

Herbrand intuition

So many models! Even for a simple formula like $P(c)$ there are infinitely many structures and models

$\mathcal{A}_1 \not\models P(c)$	$\mathcal{A}_2 \models P(c)$	$\mathcal{A}_3 \not\models P(c)$	$\mathcal{A}_4 \models P(c)$
$D_{\mathcal{A}_1} = \{a\}$ $c^{\mathcal{A}_1} = a$ $P^{\mathcal{A}_1}(a) = 0$	$D_{\mathcal{A}_2} = \{a\}$ $c^{\mathcal{A}_2} = a$ $P^{\mathcal{A}_2}(a) = 1$	$D_{\mathcal{A}_3} = \{b\}$ $c^{\mathcal{A}_3} = b$ $P^{\mathcal{A}_3}(b) = 0$	$D_{\mathcal{A}_4} = \{b\}$ $c^{\mathcal{A}_4} = b$ $P^{\mathcal{A}_4}(b) = 1$

$\mathcal{A}_5 \not\models P(c)$	$\mathcal{A}_6 \models P(c)$	$\mathcal{A}_7 \not\models P(c)$	$\mathcal{A}_8 \models P(c)$
$D_{\mathcal{A}_5} = \{a, b\}$ $c^{\mathcal{A}_5} = b$ $P^{\mathcal{A}_5}(b) = 0$ $P^{\mathcal{A}_5}(a) = 0$ (or $P^{\mathcal{A}_5}(a) = 1$)	$D_{\mathcal{A}_6} = \{a, b\}$ $c^{\mathcal{A}_6} = b$ $P^{\mathcal{A}_6}(b) = 1$ $P^{\mathcal{A}_6}(a) = 0$ (or $P^{\mathcal{A}_6}(a) = 1$)	$D_{\mathcal{A}_7} = \{a, b\}$ $c^{\mathcal{A}_7} = a$ $P^{\mathcal{A}_7}(a) = 0$ $P^{\mathcal{A}_7}(b) = 0$ (or $P^{\mathcal{A}_7}(b) = 1$)	$D_{\mathcal{A}_8} = \{a, b\}$ $c^{\mathcal{A}_8} = a$ $P^{\mathcal{A}_8}(a) = 1$ $P^{\mathcal{A}_8}(b) = 0$ (or $P^{\mathcal{A}_8}(b) = 1$)

- All structures are quite similar!
- Changing domains does not seem to change much
- The interpretation of predicates appears crucial
- The interpretation of functions appears to be isomorphic for different domains

Cardinality of domain

- Changing domains does not seem to change much
- But...

$$P(c) \wedge \neg P(d)$$

$\mathcal{A} \models P(c) \wedge \neg P(d)$

$D_{\mathcal{A}} = \{a, b\}$
$c^{\mathcal{A}} = a$
$d^{\mathcal{A}} = b$
$P^{\mathcal{A}}(a) = 1$
$P^{\mathcal{A}}(b) = 0$

- Is there a model whose domain has cardinality 1?
- No, there isn't!
- Cardinality of the domain is important

Herbrand models (1)

Herbrand interpretation — $\mathcal{H}$

- Use the set of ground terms of the formula as domain!
  - This domain is called Herbrand universe
  - **Note:** If the language contains no constant, add an arbitrary constant
- Interpret function symbols as themselves
  - $c^{\mathcal{H}} = c$ for constants
  - $f^{\mathcal{H}}(t_1, \ldots, t_n) = f(t_1^{\mathcal{H}}, \ldots, t_n^{\mathcal{H}})$

- **Note:** Interpretations of predicates is not fixed
- Herbrand base: $\{P(t_1, \ldots, t_n) \mid P$ is a predicate of arity n, $t_1, \ldots, t_n$ are ground terms $\}$
- Each predicate is a subset of the Herbrand base

An Herbrand model of a set Γ of wffs is an Herbrand interpretation satisfying Γ

Herbrand models (2)

Example

1. $N(o)$
2. $\forall x\ (N(x) \rightarrow N(s(x)))$
3. $\forall x\ \forall y\ (\neg E(x, y) \rightarrow \neg E(s(x), s(y)))$
4. $\forall x\ \neg E(s(x), o)$

- $D_{\mathcal{H}} = \{o, s(o), s(s(o)), s(s(s(o))), \ldots\}$
- $o^{\mathcal{A}} = o$
- $s^{\mathcal{A}}(o) = s(o)$; $s^{\mathcal{A}}(s(o)) = s(s(o))$; $s^{\mathcal{A}}(s(s(o))) = s(s(s(o))$; ...
- $N^{\mathcal{A}}(o) = 1$; $N^{\mathcal{A}}(s(o)) = 1$; $N^{\mathcal{A}}(s(s(o))) = 1$; ...
- $E^{\mathcal{A}}(o, o) = 1$; $E^{\mathcal{A}}(o, s(o)) = 0$; $E^{\mathcal{A}}(s(o), o) = 0$; $E^{\mathcal{A}}(s(o), s(o)) = 1$; $E^{\mathcal{A}}(s(o), s(s(o))) = 0$; ...

Theorem

Let Γ be a set of closed wffs in Skolem normal form.
Γ is satisfiable if and only if Γ has a Herbrand model.

Sketch. ($\Leftarrow$) Immediate.
($\Rightarrow$) Use structural induction. For the universal quantifier use the following lemma:

Translation Lemma

For any wff ϕ we have

$$\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(\phi[t/x]) = \nu^{(\mathcal{A},\xi^{\mathcal{A}}[\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t)/x])}(\phi)$$

Read as follows: The evaluation of $\phi[t/x]$ wrt $(\mathcal{A}, \xi^{\mathcal{A}})$ is equal to the evaluation of ϕ wrt $(\mathcal{A}, \xi^{\mathcal{A}}[\nu^{(\mathcal{A},\xi^{\mathcal{A}})}(t)/x])$

An application of the Compactness Theorem (1)

- We cannot check all Herbrand interpretations!
- We were actually interested in modeling unsatisfiability

Definition

Let $\phi = \forall x_1 \cdots \forall x_n \, \psi$ be a formula in Skolem normal form.
The Herbrand expansion of ϕ, denoted $\varepsilon(\phi)$, is the following set of ground formulas:

$$\varepsilon(\phi) = \{\psi[t_1/x_1, \ldots, t_n/x_n] \mid t_1, \ldots, t_n \in D_{\mathcal{H}}\}$$

- $\varepsilon(\phi)$ is a set of propositional wffs
- ϕ is satisfiable if and only if $\varepsilon(\phi)$ is satisfiable
- By the Compactness Theorem, $\varepsilon(\phi)$ is unsatisfiable if and only if there is a finite subset of $\varepsilon(\phi)$ that is unsatisfiable

An application of the Compactness Theorem (2)

- Let ϕ be a formula in Skolem normal form
- Fix an enumeration for the elements of $\varepsilon(\phi)$

Algorithm

1. $n := 0$
2. $n := n + 1$
3. If $\phi_1 \wedge \cdots \wedge \phi_n$ is unsatisfiable then output ϕ is unsatisfiable
4. Go to 2

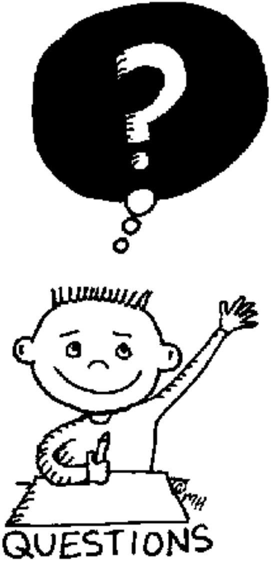
- How to check unsatisfiability of a set Γ of wffs?

Exercises (1)

1. Transform the formula

   $$\neg\exists u\,((\forall x\,P(x,u)) \rightarrow (\forall y \exists z\,(\forall v\,Q(u,v,y,z) \wedge \forall w\,Q(w,u,z,y))))$$

   into Skolem Normal Form
2. Solve Exercise 3.8, 3.9, 3.10 and 3.15 in the booklet of Ghidini and Serafini (not using the solution!)
3. Exercises from 3.16 to the end of the chapter ask to model a scenario. Have a look at them!



END OF THE LECTURE