Propositional Logic
Syntax and Semantics

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Outline

1. Syntax
   - Intuition
   - Definition
   - Convenient notation
   - Formula structure
2. Semantics
   - Meaning of a formula
   - Truth valuations
   - Interpretations
   - Models
3. Exercises

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Propositional Logic — Intuition

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Example

1. These are beans from the sack
2. These are green beans
3. These are beans from the sack **and** these are green beans

Propositions: 1, 2
Formulas (not being propositions): 3

Knowing that 1 and 2 are true we can conclude that 3 is true!

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Well-formed formulas (wff) (1)

Let V be a countable set of propositional variables

Countable means of the same cardinality of natural numbers, or smaller

Example. $V = \{A, B, C, D, A_0, A_1, \ldots\}$

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Important!

- No fixed meaning is associated to propositional variables!
- They can mean anything
- Truth value of variables fixed by semantics lateron

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The set F_V of propositional formulas (or wff) for V can defined inductively (in the next slides)

Well-formed formulas (wff) (2)

Propositional **variables** are wff

- If $v \in V$ then $v \in F_V$
- Propositional variables are called **atomic formulas**, or (propositional) **atoms**

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$\bot$ is a wff

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Well-formed formulas (wff) (3)

The negation of a wff is a wff

- If ϕ is a wff then $(\neg\phi)$ is a wff

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The **negation** of a wff is a wff

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- If $\phi \in F_V$ then $(\neg\phi) \in F_V$
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Warning! ϕ is a meta-symbol, a placeholder for a wff (not a wff itself)

Well-formed formulas (wff) (4)

The **conjunction** of two wffs is a wff

- If ϕ and ψ are wffs then $(\phi \wedge \psi)$ is a wff

Warning! ϕ and ψ are meta-symbols, and they can also be equal!

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- $\vee$ is also referred to as inclusive or

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Well-formed formulas (wff) (5)

Implication

- If ϕ and ψ are wffs then $(\phi \to \psi)$ is a wff

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- If $\phi, \psi \in F_V$ then $(\phi \to \psi) \in F_V$
- $(\phi \to \psi)$ should be true if ψ is true whenever ϕ is true

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Equivalence

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- If $\phi, \psi \in F_V$ then $(\phi \leftrightarrow \psi) \in F_V$
- $(\phi \leftrightarrow \psi)$ should be true if ψ and ϕ have the same truth value

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Well-formed formulas: Summary

$\phi \in F_V$ if and only if

- $\phi \in V$ or (atoms)
- $\phi = \top$ or (verum)
- $\phi = \bot$ or (falsum)
- $\phi = (\neg\psi)$ where $\psi \in F_V$ or (negation)
- $\phi = (\psi \wedge \psi')$ where $\psi, \psi' \in F_V$ or (conjunction)
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- $A\bot$
- $A \to \bot)$
- $A \to \neg$
- $(\to)$
- $((AB) \leftrightarrow B)$
- $((A \lor \land) \leftrightarrow \neg(\top)$
- $(\neg A \to B \to (\neg A \land C \lor B))$

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- $A \rightarrow \bot)$
- $A \rightarrow \neg$
- $(\rightarrow)$
- $((AB) \leftrightarrow B)$
- $((A \vee \wedge) \leftrightarrow \neg(\top)$
- $(\neg A \rightarrow B \rightarrow (\neg A \wedge C \vee B))$?

Equivalent definitions

Formal grammar

Terminals: $V \cup \{\top, \bot\} \cup \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\} \cup \{(,)\}$

Nonterminals: F_V

Start symbol: F_V

Production rules:

- $F_V \longrightarrow v \in V \mid \top \mid \bot$
- $F_V \longrightarrow (\neg F_V)$
- $F_V \longrightarrow (F_V \wedge F_V)$
- $F_V \longrightarrow (F_V \vee F_V)$
- $F_V \longrightarrow (F_V \rightarrow F_V)$
- $F_V \longrightarrow (F_V \leftrightarrow F_V)$

Equivalent definitions

Formal grammar

Terminals: $V \cup \{\top, \bot\} \cup \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\} \cup \{(,)\}$

Nonterminals: F_V

Start symbol: F_V

Production rules:

- $F_V \longrightarrow v \in V \mid \top \mid \bot$
- $F_V \longrightarrow (\neg F_V)$
- $F_V \longrightarrow (F_V \wedge F_V)$
- $F_V \longrightarrow (F_V \vee F_V)$
- $F_V \longrightarrow (F_V \rightarrow F_V)$
- $F_V \longrightarrow (F_V \leftrightarrow F_V)$

Language elements

- V: propositions
- $\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$: logical connectives
- **(,)**: auxiliary symbols *(parentheses, not comma)*

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Eliminating parentheses

- Too many parentheses!!!
- Can we omit a few of them?
- Let us agree on a precedence of connectives
  (also known as *binding strength*)

Usual assumption

$\neg$ is stronger than
$\wedge$ is stronger than
$\vee$ is stronger than
$\rightarrow$ is stronger than
$\leftrightarrow$

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Minimal parentheses

1. $A \to \bot$

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1. $(A \rightarrow \bot)$
2. $A \rightarrow A$

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3. $A \vee B \leftrightarrow B \vee A$

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3. $((A \vee B) \leftrightarrow (B \vee A))$
4. $A \vee B \leftrightarrow \neg(\top \rightarrow A \wedge B)$

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5. $\neg A \vee B \rightarrow \neg A \wedge C$

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Formulas as trees

Every wff can be written as a formula tree

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$$\neg A \vee B \rightarrow \neg A \wedge C$$

or equivalently

$$(((\neg A) \vee B) \rightarrow ((\neg A) \wedge C))$$

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$\rightarrow$

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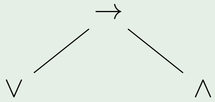

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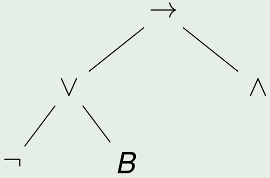

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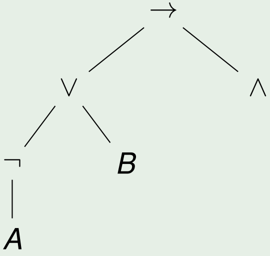

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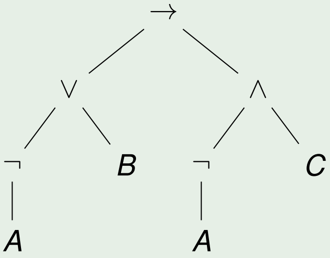

Subformulas (1)

Immediate subformulas of a wff ϕ, denoted *isf*(ϕ)

- If $\phi \in V \cup \{\top, \bot\}$ then *isf*$(\phi) = \emptyset$
- If $\phi = \neg\psi$ then *isf*$(\phi) = \{\psi\}$
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Subformulas of a wff ϕ, denoted *sf*(ϕ)

Inductive definition

1. ϕ itself belong to *sf*(ϕ)
2. If $\psi \in$ *sf*(ϕ) then *isf*$(\psi) \subseteq$ *sf*(ϕ)

sf(ϕ) is the minimal set satisfying conditions 1 and 2

Subformulas (2)

Example

Consider

$$\neg A \vee B \to \neg A \wedge C$$

Its subformulas are the following:

Subformulas (2)

Example

Consider

$$\neg A \lor B \to \neg A \land C$$

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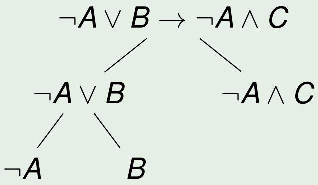

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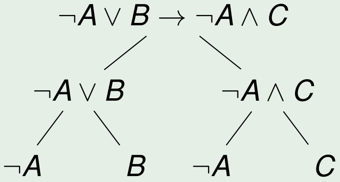

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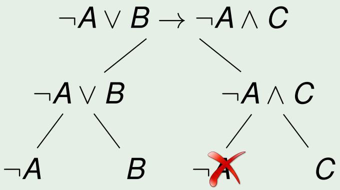

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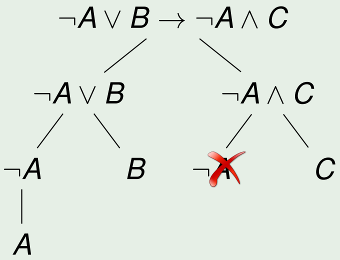

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Associate a **meaning** to wffs in a **formal** way

- We could associate “sentences” to variables (e.g. “It is raining.”)

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Associate truth values to atoms, truth values for formulas follow

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Truth valuations (1)

- Truth values are 1 (true) and 0 (false)
- A (truth) valuation ν is a function

$$\nu\colon V \mapsto \{0, 1\}$$

  - So for each $A \in V$, either $\nu(A) = 1$ or $\nu(A) = 0$, not both

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Simple formulas

- $\nu^*(\top) = 1$
- $\nu^*(\bot) = 0$
- If $A \in V$ then $\nu^*(A) = \nu(A)$

Truth valuations (2)

ϕ	ψ	$\neg\phi$
0	0	1
1	0	0
0	1	1
1	1	0

Negation: $\neg\phi$

Should always have the opposite truth value of ϕ

$\nu^*(\neg\phi) = 1$ if and only if $\nu^*(\phi) = 0$

Truth valuations (2)

ϕ	ψ	$\neg\phi$	$\phi \wedge \psi$
0	0	1	0
1	0	0	0
0	1	1	0
1	1	0	1

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Should be true if both ϕ and ψ are true

$\nu^*(\phi \wedge \psi) = 1$ iff $\nu^*(\phi) = 1$ and $\nu^*(\psi) = 1$

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Disjunction: $\phi \vee \psi$

Should be true if ϕ or ψ are true

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Truth valuations (2)

ϕ	ψ	$\neg\phi$	$\phi \wedge \psi$	$\phi \vee \psi$	$\phi \rightarrow \psi$
0	0	1	0	0	1
1	0	0	0	1	0
0	1	1	0	1	1
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Equivalence: $\phi \leftrightarrow \psi$

Should be true if ϕ has the same truth value as ψ

$\nu^*(\phi \leftrightarrow \psi) = 1$ iff $\nu^*(\phi) = \nu^*(\psi)$

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Interpretations

An interpretation *I* consists exactly of a truth valuation ν

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Given a wff ϕ and an interpretation I consisting of valuation ν, let

$$I(\phi) = \nu^*(\phi)$$

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- The truth value is the meaning of the formula

Interpretations

An interpretation I consists exactly of a truth valuation ν

Given a wff ϕ and an interpretation I consisting of valuation ν, let

$$I(\phi) = \nu^*(\phi)$$

- I associates a unique truth value to every formula
- The truth value is the meaning of the formula

I is usually represented as the set of variables interpreted as true:

$$I = \{A \in V \mid I(A) = 1\}$$

Calculating truth values of formulas (1)

Given a formula ϕ and interpretation *I*, **how** to determine $I(\phi)$?

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Example (For interpretation $I = \{C, A\}$)

Consider

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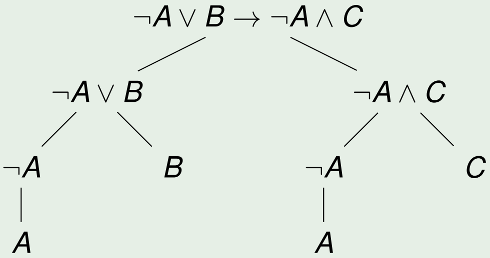
Its subformulas are the following:



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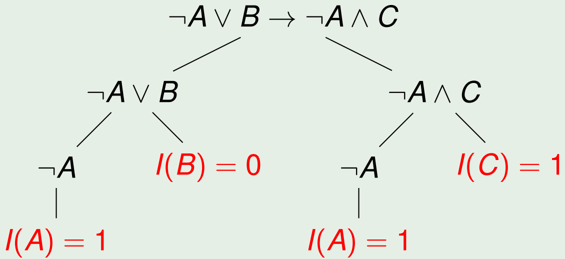

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$$\neg A \vee B \qquad \neg A \wedge C$$

$$I(\neg A) = 0 \quad I(B) = 0 \quad I(\neg A) = 0 \quad I(C) = 1$$

$$I(A) = 1 \qquad I(A) = 1$$

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$$I(A) = 1 \qquad I(A) = 1$$

Calculating truth values of formulas (2)

A truth table may be useful for calculating truth values for more than one interpretation

Example

Consider

$$\neg A \lor B \to \neg A \land C$$

A	B	C	$\neg A$	$\neg A \lor B$	$\neg A \land C$	$\neg A \lor B \to \neg A \land C$

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0	0	0	1	1	0	0

Calculating truth values of formulas (2)

A truth table may be useful for calculating truth values for more than one interpretation

Example

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A	B	C	$\neg A$	$\neg A \vee B$	$\neg A \wedge C$	$\neg A \vee B \rightarrow \neg A \wedge C$
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
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1	0	1	0	0	0	1
1	1	0	0	1	0	0
1	1	1	0	1	0	0

Outline

1 Syntax
- Intuition
- Definition
- Convenient notation
- Formula structure

2 Semantics
- Meaning of a formula
- Truth valuations
- Interpretations
- Models

3 Exercises

Models

1. An interpretation *I* is a model of a wff ϕ if $I(\phi) = 1$
2. If $I(\phi) = 1$ then *I* satisifies ϕ
3. If *I* satisfies ϕ then we write $I \models \phi$

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Let $V = \{x, y, z\}$ be a set of propositional variables.
Show whether the following are well-formed formulas (*not* considering operator preferences) in F_V or not:

1. $(\neg(z \wedge x))$
2. $((x \leftarrow y) \wedge z)$
3. $((x \leftrightarrow (\neg y)))$
4. $(\neg((\neg(\neg z)) \vee (\neg x)))$
5. $((x \leftrightarrow y) \wedge (\neg\neg x))$
6. $((y \wedge x) \leftrightarrow (w \wedge (\neg z)))$
7. $((x \rightarrow (\neg x)) \vee (\neg(y) \rightarrow (\neg z)))$

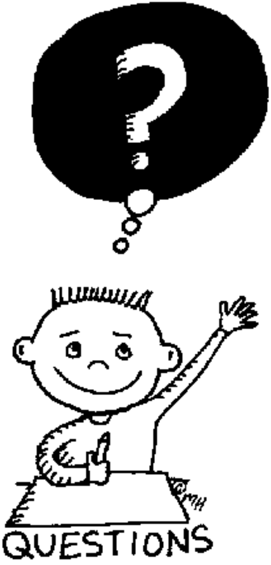
Let $V = \{x, y, z\}$ be a set of propositional variables.
Draw the parse tree and simplify (according to the standard preferences) as much as possible each of the following formulas:

1. $(\neg(y \wedge (\neg(z \rightarrow (x \vee (\neg y))))))$
2. $(((\neg x) \vee (\neg y)) \leftrightarrow ((\neg y) \wedge (\neg z)))$
3. $((x \rightarrow (\neg y)) \leftrightarrow (\neg(y \wedge (\neg z))))$
4. $((\neg x) \wedge ((\neg y) \rightarrow (\neg(z \vee (\neg y)))))$

Let $V = \{x, y, z\}$ be a set of propositional variables.
Find the full version (according to the standard preferences) and draw the parse trees of the following formulas:

1. $x \lor \neg y \land \neg z$
2. $\neg x \lor z \leftrightarrow y \land \neg z$
3. $\neg x \leftrightarrow z \to \neg y \land z$
4. $x \leftrightarrow \neg y \to \neg z \land \neg x \lor y$

1. List all subformulas of formulas
   - $x \vee \neg y \wedge \neg z$
   - $\neg x \vee z \leftrightarrow y \wedge \neg z$
2. Given the interpretation $I = \{x, z\}$, decide whether I is a model of
$$x \vee \neg y \wedge \neg z$$
3. Work out the truth table of formula
$$((\neg x) \wedge ((\neg y) \rightarrow (\neg(z \vee (\neg y)))))$$



END OF THE LECTURE