Propositional Logic
Sequent Calculus

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Outline

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Intuition

Idea

Define inference rules for sequents

$$\Gamma \vdash \Delta$$

where Γ and Δ are sequences of formulas

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Goal 1 $\Gamma \vdash \Delta$ holds if $\Gamma \models \bigvee \Delta$ (completeness)
Goal 2 $\Gamma \vdash \Delta$ implies $\Gamma \models \bigvee \Delta$ (soundness)

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Notation

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S}$$

From sequents S_1 (and S_2) conclude sequent S.

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System considered here: **LK**, defined by Gerhard Gentzen

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Axioms

Structural rules

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$$\frac{}{A \vdash A}\,(ax)$$

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LK — Logistischer Klassischer Kalkül (1)

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permutation $\quad \dfrac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,(p.l) \quad \dfrac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,(p.r)$

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contraction

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(c.l) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,(c.r)$$

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weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(w.l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,(w.r)$$

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cut

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(cut)$$

Axioms

$$\frac{}{A \vdash A}\,(ax)$$

What are A and B?
What $\Gamma, \Gamma', \Delta, \Delta'$?

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$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\,(\vee r.2)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\,(\neg l) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,(\neg r)$$

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$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to l) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\,(\to r)$$

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Derivation

- Use these inference rules consecutively

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Example

$$\dfrac{\dfrac{}{A \vdash A}\,(ax)}{\vdash A, \neg A}\,(\neg r)$$

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- If on top there are only axioms then it is a derivation of the bottom sequent

Derivation

- Use these inference rules consecutively

Example

$$\dfrac{\dfrac{}{A \vdash A}\,(ax)}{\vdash A, \neg A}\,(\neg r)$$

- If on top there are only axioms then it is a derivation of the bottom sequent

Theorem

Sequent Calculus is sound and complete, i.e.,
if we can derive $\Gamma \vdash \Delta$ then $\Gamma \models \bigvee \Delta$, and
if $\Gamma \models \bigvee \Delta$ then there is a derivation for $\Gamma \vdash \Delta$.

Outline

LK — Summary

$$\frac{}{A \vdash A}(ax) \qquad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}(p.l) \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}(p.r) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}(w.l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}(w.r)$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}(c.l) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}(c.r) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}(cut)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}(\wedge l.1) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}(\wedge r)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}(\wedge l.2)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}(\vee l) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}(\vee r.1)$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}(\vee r.2)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}(\neg l) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}(\neg r)$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}(\to l) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}(\to r)$$

LK — Example

Example

Do the following entailments hold?

1. $(A \vee \neg B) \wedge B, \neg A \models \bot$
2. $A \wedge C, \neg A \vee B \models \bot$

(On the blackboard.)

Outline

Exercises (1)

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

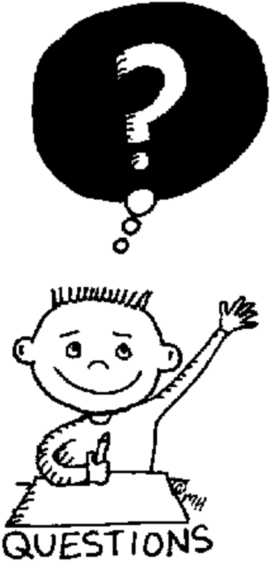
Give proof trees for the following tautologies:

1. $A \to (B \to A)$
2. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
3. $A \to (B \to A \wedge B)$
4. $A \to A \vee B$
5. $B \to A \vee B$
6. $(A \to B) \to ((A \to \neg B) \to \neg A)$
7. $A \wedge B \to A$
8. $A \wedge B \to B$
9. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
10. $\neg\neg A \to A$

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

Give proof trees for the following equivalences:

1. $(A \vee B) \vee C \equiv A \vee (B \vee C)$ (associativity)
2. $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ (associativity)
3. $A \vee B \equiv B \vee A$ (commutativity)
4. $A \wedge B \equiv B \wedge A$ (commutativity)
5. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (distributivity)
6. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (distributivity)
7. $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan)
8. $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan)
9. $A \vee A \equiv A$ (idempotency)
10. $A \wedge A \equiv A$ (idempotency)
11. $\neg\neg A \equiv A$ (double negation)
12. $(A \vee B) \wedge (\neg A \vee C) \equiv (A \vee B) \wedge (\neg A \vee C) \wedge (B \vee C)$ (resolution)



END OF THE LECTURE